Abelian extensions of global fields with constant local degrees

Hershy Kisilevsky, Jack Sonn*

Mathematics Department, Concordia University, Montreal, Canada; Mathematics Department, Technion, Haifa, Israel

2000 Mathematics Subject Classification. 11K

Let $K$ be a field, $Br(K)$ its Brauer group. If $L/K$ is a field extension, then the relative Brauer group $Br(L/K)$ is the kernel of the restriction map $\text{res}_{L/K} : Br(K) \to Br(L)$. Relative Brauer groups have been studied by Fein and Schacher. Every subgroup of $Br(K)$ is a relative Brauer group $Br(L/K)$ for some extension $L/K$, and the question arises as to which subgroups of $Br(K)$ are algebraic relative Brauer groups, i.e. of the form $Br(L/K)$ with $L/K$ an algebraic extension. For example if $L/K$ is a finite extension of number fields, then $Br(L/K)$ is infinite, so no finite subgroup of $Br(K)$ is an algebraic relative Brauer group. In [1] the question was raised as to whether or not the $n$-torsion subgroup $Br_n(K)$ of the Brauer group $Br(K)$ of a field $K$ is an algebraic relative Brauer group. For example, if $K$ is a ($p$-adic) local field, then $Br(K) \cong \mathbb{Q}/\mathbb{Z}$, so $Br_n(K)$ is an algebraic relative Brauer group for all $n$. A counterexample was given in [1] for $n = 2$ and $K$ a formal power series field over a local field. For global fields $K$, the problem is a purely arithmetic one, because of the fundamental local-global description of the Brauer group of a global field. In particular, for a Galois extension $L/K$ of global fields, if the local degree of $L/K$ at every finite prime is equal to $n$, and is equal to 2 at the real primes for $n$ even, then $Br(L/K) = Br_n(K)$. In [1], it was proved that $Br_n(\mathbb{Q})$ is an algebraic relative Brauer group for all squarefree $n$. In [2], the arithmetic criterion above was verified for any number field $K$ Galois over $\mathbb{Q}$ and any $n$ prime to the class number of $K$, so in particular, $Br_n(\mathbb{Q})$ is an algebraic relative Brauer group for all $n$. In [3], Popescu proved that for a global function field $K$ of characteristic $p$, the arithmetic criterion holds for $n$ prime to the order of the non-$p$ part of the Picard group of $K$. In this paper we settle the question completely, by verifying the arithmetic criterion for all $n$ and all global fields $K$. In particular, the $n$-torsion subgroup of the Brauer group of $K$ is an algebraic relative Brauer group for all $n$ and all global fields $K$. The proof, an extension of the ideas in [2], reduces to the case $n$ a prime power $\ell^r$. We first carry out the proof for number fields $K$. The proof for the function field case when $\ell \neq \text{char}(K)$ is essentially the same as the proof in the number field case. The proof for $\ell = \text{char}(K)$ appears in [3].