Fibonacci numbers and orthogonal polynomials

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Let \( F_0 = 0, F_1 = 0, \ldots \) with \( F_{n+1} = F_n + F_{n-1}, n \geq 1 \) be the sequence of Fibonacci numbers. We prove that \( (1/F_{n+2}) \) is the moment sequence of a discrete probability and we identify the corresponding orthogonal polynomials as little \( q \)-Jacobi polynomials, cf. [1]

\[
p_n(x\phi; q, 1; q) = 2\varphi_1 \left( \frac{q^{-n}, q^{n+2}}{q^2}; q, -\frac{x}{\phi} \right),
\]

where \( \phi = (1+\sqrt{5})/2 \) is the golden ratio and \( q = -1/\phi^2 \). The corresponding Hankel matrix \( (1/F_{i+j+2}) \) was called the Filbert matrix in [2], and it was established via computer algebra that its inverse matrix has integer coefficients expressed in terms of the Fibonomial coefficients

\[
\binom{n}{k}_F = \prod_{i=1}^{k} \frac{F_{n-i+1}}{F_i}, \quad 0 \leq k \leq n.
\]

We prove that

\[
F_{n+1}p_n(x\phi; q, 1; q) = \sum_{k=0}^{n} (-1)^{kn-k} \binom{n}{k}_F \binom{n+k+1}{n}_F x^k,
\]

and that the corresponding kernel polynomials for the orthonormal polynomials have integer coefficients. This explains the result of Richardson. A similar but more elementary result holds for the Hilbert matrix \( (1/(i+j+1)) \), which is the Hankel matrix for Lebesgue measure on \([0,1]\), and the corresponding orthogonal polynomials are Legendre polynomials.