On the prime geodesic theorem

Muharem Avdispahic* and Lejla Smajlovic

Department of Mathematics, University of Sarajevo, Zmaja od Bosne 35, 71 000
Sarajevo, Bosnia and Herzegovina [mavdispa@pmf.unsa.ba]

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Let $\Gamma \subseteq \text{PSL}(2, \mathbb{R})$ be a Fuchsian group of the first kind containing $n_1$ inequivalent parabolic elements and let $W$ be a unitary $r \times r$ multiplier system of a weight $m$ on $\Gamma$. The Selberg zeta function associated to the pair $(\Gamma, W)$ is defined as an Euler product over all primitive hyperbolic conjugacy classes

$$Z_{\Gamma, W}(s) = \prod_{\{P_0\}_\Gamma} \prod_{k=0}^{\infty} \det \left( I_r - W(P_0) N(P_0)^{-s-k} \right),$$

converging absolutely for $\Re s > \max \left\{ 1, \frac{1}{2} + \frac{|m|}{2} \right\}$.

The prime geodesic counting function is defined by

$$\pi_0(x, W) = \sum_{N(P_0) \leq x} \text{Tr} W(P_0), \quad x \geq 1.$$

Extending an explicit formula for a triple $(Z, Z, \Psi)$ in the Jorgenson-Lang fundamental class of functions, we obtain an improved bound for the logarithmic derivative of the Selberg zeta function in the critical strip:

$$\frac{Z'_{\Gamma, W}}{Z_{\Gamma, W}} \left( \frac{1}{2} + \alpha \right) = O \left( \min \left\{ \frac{T}{\sigma \log |T|}, \frac{T^{1-2\sigma}}{\sigma} \right\} \right) \quad \text{as} \quad |T| \to \infty,$$

for $\alpha = \sigma + iT$, $\frac{1}{2} > \sigma > 0$.

Let $0 = \lambda_0 < \lambda_1 < ...$, $\lambda_n \to \infty$, be the discrete spectrum of the unique self-adjoint extension of the operator $-\Delta_m = -y^2 \left( \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right) + im\pi \frac{\partial}{\partial x}$. An application of the above bound for the logarithmic derivative, yields the prime geodesic theorem in the following form.

**Theorem.** For $x \geq 2$, $\pi_0(x, W) = \sum_{n=0}^{K} li \left( x^{s_n} \right) + O \left( x^{\frac{3}{4}} (\log x)^{-1} \right)$

where $s_n = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_n}$, $K$ is the number of $\lambda_n \in \left[ 0, \frac{1}{4} \right)$ and the implied constant depends only on $\Gamma$, $m$ and $W$.