Stabilizers and orbits of smooth functions

Sergiy Maksymenko

Topology dept., Institute of Mathematics, NAS of Ukraine, Tereshchenkiivs'ka str., 3, Kyiv, 01601, Ukraine [maks@imath.kiev.ua]

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Let $M$ be a smooth connected compact manifold, $f : M \to \mathbb{R}^1$ a smooth function, $D_M$ the group of diffeomorphisms of $M$, and $D_{R^1}(f)$ the group of preserving orientation diffeomorphisms of $M$ that leave invariant the image $f(M)$ of $f$. There are two natural actions of the groups $D_{R^1}(f) \times D_M$ and $D_M$ on $C^\infty(M, \mathbb{R}^1)$:

$D_{R^1}(f) \times D_M \times C^\infty(M, \mathbb{R}^1) \to C^\infty(M, \mathbb{R}^1)$ \quad (\phi, h) \cdot f = \phi \circ f \circ h^{-1},$

$D_M \times C^\infty(M, \mathbb{R}^1) \to C^\infty(M, \mathbb{R}^1)$ \quad h \cdot f = f \circ h^{-1},

where $\phi \in D_{R^1}(f)$ and $h \in D_M$. Let $S_{MR^1}$ and $S_M$ be corresponding stabilizers and $O_{MR^1}$ and $O_M$ the corresponding orbits of $f$ with respect to these actions. We endow all these spaces with the corresponding $C^\infty$ Whitney topologies.

For $z \in M$ let $C_z^\infty(M)$ be the algebra of germs of smooth functions at $z$ and let $\Delta(f, z)$ be the Jacobi ideal in $C_z^\infty(M)$ generated by germs of partial derivatives of $f$ at $z$. We put on $f$ the following three conditions:

1. $f$ is constant at every connected component of $\partial M$;
2. there are only finitely many critical values of $f$;
3. for every critical point $z$ of $f$ the germ of the function $f(x) - f(z)$ belongs to $\Delta(f, z)$, i.e. there is a vector field $F$ near $z$ such that $f(x) - f(z) = df(F)(x)$.

Condition (3) holds for a very large class of singularities. In particular non-degenerate and simple singularities and even formal series satisfy (3). It is also preserved by stable equivalence of singularities.

Let $D_{R^1}(f)$ be the subgroup of $D_{R^1}(f)$ consisting of diffeomorphisms that also fix every boundary or critical value of $f$.

**Definition.** Say that a critical point $z$ of $f$ is essential if for every neighborhood $U_z$ of $z$ there exists a neighborhood $V_f$ of $f$ in $C^\infty(M)$ with $C^\infty$-topology such that every $g \in V_f$ has a critical point in $U_z$. E.g. if $f(x) = x^2$ and $g(x) = x^3$, then $0 \in \mathbb{R}^1$ is essential for $f$ but not for $g$.

**Theorem** \(\text{(A) Suppose that } f \text{ satisfies (1)-(3). Then we have an exact sequence } 1 \to S_M \to S_{MR^1} \to D_{R^1}(f) \to 1, \text{ i.e. the stabilizer } S_{MR^1} \text{ is an extension of } S_M \text{ by } D_{R^1}(f). \text{ This sequence split by a homomorphism } \theta : D_{R^1}(f) \to S_{MR^1}, \text{ whence the embedding } S_M \times \text{id}_M \subset S_{MR^1} \text{ extends to a homeomorphism between } S_M \times D_{R^1}(f) \text{ and } S_{MR^1}. \text{ In particular, } S_M \times \text{id}_M \text{ is a strong deformation retract of } S_{MR^1}.\)

(B) In addition, suppose that every critical level-set of $f$ includes either an essential critical point or a connected component of $\partial M$. Then the embedding $O_M \subset O_{MR^1}$ extends to a homeomorphism $O_M \times \mathbb{R}^{n-2} \approx O_{MR^1}$, where $n$ is the total number of critical and boundary values of $f$.