Fréchet differentiability for an optimal control problem of temperature in thin fabric sheets

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In paper [1] was considered situation when a moving sheet of glass fabric was heated up in a furnace in order to burn out oil. The main concern there was to find the best way of cooling of the fabric sheet after it had been heated up and oil had been burned out.

We consider a simplified situation from [1], when oil burnout is neglected and process is steady in time. Our aim is to find the optimal (in some sense) temperature distribution $T$ in fabric sheet $\Omega = [-l_1, l_1] \times [-l_2, l_2] \times [-\delta, \delta]$ by varying temperature $T_{ht}$ on the heaters $\Sigma_{ht}$ of the furnace. As simultaneous conductive-radiative heat transfer occurs in the furnace, then this leads to an optimal control problem for an elliptic boundary value problem with an integral equation defined on the boundary (2):

As thickness $2\delta$ of the sheet is small, temperature $T(x_1, x_2, x_3)$ can be approximated by a function $\tilde{T}(x_1, x_2)$ and the original optimal control problem can be replaced by a simpler one:

$$I(\tilde{T}) \mapsto \min,$$

$$G_1(||\tilde{T}||^3) + k\tilde{T} = G_2(||T_{ht}||^3T_{ht}) + kg,$$

where $G_1 : L_\infty(Q) \mapsto L_\infty(Q)$, $G_2 : L_\infty(\Sigma_{ht}) \mapsto L_\infty(Q)$ are linear bounded operators, $g \in L_\infty(Q)$ and $Q = [-l_1, l_1] \times [-l_2, l_2]$.

The main result, that we want to present here, is that under certain conditions the last equation have one and only one solution $\Phi(T_{ht}) \in L_\infty(Q)$ for every fixed $T_{ht} \in L_\infty(\Sigma_{ht})$. Moreover, the mapping $T_{ht} \mapsto \Phi(T_{ht})$ from $L_\infty(\Sigma_{ht})$ to $L_\infty(Q)$ is Fréchet differentiable and therefore the standard formula for increment of the cost functional holds:

$$I(\Phi(T_{ht}^2)) - I(\Phi(T_{ht}^1)) = L[T_{ht}^1](T_{ht}^2 - T_{ht}^1) + o(||T_{ht}^2 - T_{ht}^1||_{L_\infty(\Sigma_{ht})}),$$

where $T_{ht}^1 \in L_\infty(\Sigma_{ht})$, $T_{ht}^2 \in L_\infty(\Sigma_{ht})$, $L[T_{ht}^1] \in L_\infty^*(\Sigma_{ht})$.
