Asymptotic solutions for large time of Hamilton–Jacobi equations

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Abstract. In this article we discuss some recent results on the large-time behavior of solutions of Hamilton–Jacobi equations as well as some ideas and observations behind them and historical remarks concerning them.


Keywords. Large-time behavior, asymptotic solutions, Hamilton–Jacobi equations, Aubry sets, weak KAM theory, viscosity solutions.

1. Introduction

Hamilton–Jacobi equations play important roles in classical mechanics, geometric optics, optimal control, differential games, etc. We are here interested in global solutions of Hamilton–Jacobi equations. A well-known classical method of finding solutions of Hamilton–Jacobi equations is that of characteristics and its applications have serious difficulties in practice because of developments of shocks in solutions. At the beginning of 1980s M. G. Crandall and P.-L. Lions [10], [11] introduced the notion of viscosity solution in the study of Hamilton–Jacobi equations. It is a notion of generalized solutions for partial differential equations and it is based on the maximum principle while, in this regard, distributions theory is based on integration by parts. This notion has been successfully employed to study fully nonlinear partial differential equations (PDE for short), especially Hamilton–Jacobi equations and fully nonlinear elliptic or parabolic PDE. Important basic features of viscosity solutions are: they enjoy nice properties such as (1) stability under uniform convergence or under the processes of pointwise supremum or infimum, and (2) existence and uniqueness of solutions, under mild assumptions, of boundary value problems or the Cauchy problem for fully nonlinear PDE. See [2], [3], [24], [9] for general overviews and developments of the theory of viscosity solutions.

We recall the definition of viscosity solutions of $F(x, u(x), Du(x)) = 0$ in $\Omega$. Let $u \in C(\Omega, \mathbb{R})$. It is called a viscosity subsolution (resp., supersolution) of $F[u] = 0$ in $\Omega$ if whenever $\varphi \in C^1(\Omega)$ and $u-\varphi$ attains a maximum (resp., minimum) at $y \in \Omega$, then $F(y, u(y), D\varphi(y)) \leq 0$ (resp., $F(y, u(y), D\varphi(y)) \geq 0$). Then, $u \in C(\Omega)$

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is called a viscosity solution of $F[u] = 0$ in $\Omega$ if it is both a viscosity sub- and supersolution of $F[u] = 0$ in $\Omega$. We will be here focused on viscosity solutions, subsolutions, or supersolutions and will suppress the term “viscosity” in what follows as far as there is no danger of confusion.

In this article we consider the stationary Hamilton–Jacobi equation

$$H(x, Du) = 0 \quad \text{in } \Omega,$$

(1)

where $\Omega$ is an open subset of $\mathbb{R}^n$, and the Cauchy problem

$$u_t + H(x, Du) = 0 \quad \text{in } \Omega \times (0, \infty),$$

(2)

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega.$$

(3)

Here $H$ is a continuous function on $\bar{\Omega} \times \mathbb{R}^n$ and $u$ represents the real-valued unknown function on $\Omega$ in the case of (1) or on $\bar{\Omega} \times [0, \infty)$ in the case of (2) and (3), respectively. We write frequently $H[u]$ for $H(x, Du(x))$ for notational simplicity. We will be concerned also with the additive eigenvalue problem

$$H[v] = c \quad \text{in } \Omega.$$

(4)

Here the unknown is a pair $(c, v)$ of a constant $c \in \mathbb{R}$ and a function $v$ on $\Omega$ for which $v$ satisfies (4).

The purpose of this article is to review some of recent results concerning the large-time behavior of solutions of (2). An interesting feature of the investigations towards such results is the interaction with the developments of weak KAM theory, and this review will touch upon weak KAM theory. For overviews and developments of weak KAM theory, we refer to [17], [14].

In Section 2 we discuss projected Aubry sets and representation formulas for solutions of (1). In Section 3 the main result concerning the large-time behavior of solutions of (2) are explained. In Section 4 we outline the proof of the main result.

### 2. Projected Aubry sets and representation of solutions

Weak KAM theory introduced by A. Fathi in [15], [17] has changed the viewpoint of uniqueness questions regarding (1).

To begin with, we recall that classical uniqueness or comparison results in viscosity solutions theory applied to the following simple PDE

$$\lambda u + H(x, Du) = 0 \quad \text{in } \Omega,$$

(5)

where $\lambda$ is a positive constant, states:

**Theorem 2.1.** Let $u, v \in C(\bar{\Omega})$ be a subsolution and a supersolution of (5), respectively. Assume that $\Omega$ is bounded, that either $u$ or $v$ is locally Lipschitz in $\Omega$, and that $u \leq v$ on $\partial \Omega$. Then $u \leq v$ in $\Omega$.
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See [2], [3], [8] for general comparison results for Hamilton–Jacobi equations. The above theorem guarantees uniqueness of locally Lipschitz continuous solutions of the Dirichlet problem for (5). We will be concerned mostly with viscosity sub-, super-, or solutions of (1) which are locally Lipschitz continuous, and thus the assumption concerning local Lipschitz continuity of solutions in the above theorem is not any real restriction.

Another way of stating the above theorem is as follows.

**Theorem 2.2.** Let \( u, v \in C(\bar{\Omega}) \) be solutions, respectively, of \( H[u] \leq -\varepsilon \) in \( \Omega \) and of \( H[v] \geq 0 \) in \( \Omega \), where \( \varepsilon \) is a positive constant. Assume that \( \Omega \) is bounded, that either \( u \) or \( v \) is locally Lipschitz in \( \Omega \), and that \( u \leq v \) on \( \partial \Omega \). Then \( u \leq v \) in \( \Omega \).

Let \( \Omega = \text{int} B(0, r) \), where \( B(0, r) \) denotes the closed ball of radius \( r > 0 \) with center at the origin and \( \text{int} A \) denotes the interior of \( A \subset \mathbb{R}^n \). The eikonal equation \( |Du| = |x| \) in \( \Omega \) has two solutions \( u_\pm(x) := \pm(1 - |x|^2)/2 \), which in addition satisfies the boundary condition, \( u(x) = 0 \) on \( \partial \Omega \). Indeed, the solutions of the Dirichlet problem, \( |Du| = |x| \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), are given by the family of functions \( u_a(x) := \min\{u_+(x), a + u_-(x)\} \) parametrized by \( a \in [0, 1] \). This example tells us that the Dirichlet problem, \( |Du| = |x| \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), has many solutions and that the condition, \( \lambda > 0 \), in Theorem 2.1 is sharp. Thus uniqueness of solution of the Dirichlet problem does not hold in general for (1).

We assume henceforth the following two assumptions, the convexity and coercivity of the Hamiltonian \( H \):

\[
\text{for each } x \in \Omega \text{ the function } p \mapsto H(x, p) \text{ is convex in } \mathbb{R}^n, \quad (6)
\]

\[
\lim_{r \to \infty} \inf_{x \in \Omega, \ p \in \mathbb{R}^n \setminus B(0, r)} H(x, p) = \infty. \quad (7)
\]

We set \( L(x, \xi) = \sup_{p \in \mathbb{R}^n} (\xi \cdot p - H(x, p)) \) for \( (x, \xi) \in \Omega \times \mathbb{R}^n \), where \( \xi \cdot p \) denotes the Euclidean inner product of \( \xi, \ p \in \mathbb{R}^n \). We call the function \( L: \Omega \times \mathbb{R}^n \to (-\infty, \infty] \) the **Lagrangian**.

We define the function \( d_H \) on \( \Sigma \times \Omega \) by

\[
d_H(x, y) = \sup\{v(x) \mid H[v] \leq 0, \ v(y) = 0\}.
\]

Classical results in viscosity solutions theory assure that the function \( d_H \) has the properties:

\[
H[d(\cdot, y)] \leq 0 \quad \text{in } \Omega, \quad (8)
\]

\[
H[d(\cdot, y)] \geq 0 \quad \text{in } \Omega \setminus \{y\}. \quad (9)
\]

Following [18], we define the (projected) **Aubry set** \( \mathcal{A} \) for the Lagrangian \( L \) (or for the Hamiltonian \( H \)) as the subset of \( \Omega \) given by

\[
\mathcal{A} = \{y \in \Omega \mid H[d(\cdot, y)] \geq 0 \text{ in } \Omega\}. \quad (10)
\]
In view of (8), (9), and (10), it is easily seen that \( y \in \Omega \setminus \mathcal{A} \) if and only if \( H(y, p) < 0 \) for some \( p \in D_{\mathcal{A}}^-d_H(y, y) \), where \( D_{\mathcal{A}}^-d_H(x, y) \) denotes the subdifferential of \( d_H(\cdot, y) \) at \( x \). Similarly, we may state that \( y \in \Omega \setminus \mathcal{A} \) if and only if there are functions \( \sigma \in C(\Omega) \) and \( \psi \in C_{0+}^{0+}(\Omega) \) such that \( \sigma \geq 0 \) in \( \Omega \), \( \sigma(y) > 0 \), and \( H[\psi] \leq -\sigma \) in \( \Omega \).

We now assume for simplicity of presentation that \( \Omega \) is an \( n \)-dimensional torus \( \mathbb{T}^n \).

**Theorem 2.3.** Let \( u, v \in C(\Omega) \) be a subsolution and a supersolution of \( H = 0 \) in \( \Omega \), respectively. Assume that \( u \leq v \) on \( \mathcal{A} \). Then \( u \leq v \) in \( \Omega \).

This theorem can be found in [17, Chap. 8]. A key observation for the proof of the above theorem is that for each compact \( K \subset \Omega \setminus \mathcal{A} \) there exist a function \( \psi_K \in C_{0+}^{0+}(\Omega) \) and a constant \( \varepsilon_K > 0 \) such that \( H[\psi_K] \leq -\varepsilon_K \) in a neighborhood \( V_K \) of \( K \).

Indeed, with such \( K, \psi_K, \varepsilon_K \), and \( V_K \), we see that for any \( \lambda \in (0, 1) \), the function \( u_\lambda := (1 - \lambda)u + \lambda \psi_K \) is a subsolution of \( H[u_\lambda] \leq -\lambda \varepsilon_K \) in \( V_K \), and hence from Theorem 2.2 that for all \( x \in V_K \),

\[
(1 - \lambda)u(x) + \lambda \psi_K(x) \leq v(x) + \sup_{\Omega \setminus K} ((1 - \lambda)u + \lambda \psi_K - v),
\]

which implies that for all \( x \in \Omega \setminus \mathcal{A} \),

\[
u(x) \leq v(x) + \sup_{\mathcal{A}} (u - v).
\]

In their study of semicontinuous viscosity solutions, E. N. Barron and R. Jensen [4], [5] have observed that under the convexity assumption (6), a function \( u \in C(\Omega) \) is a viscosity subsolution of (1) if and only if \( H(x, p) \leq 0 \) for all \( p \in D^-u(x) \) and \( x \in \Omega \), where \( D^-u(x) \) denotes the subdifferential of \( u \) at \( x \). A consequence of this observation is that the pointwise infimum of a uniformly bounded family of solutions of (1) yields a solution of (1).

**Theorem 2.4.** If \( u \in C(\Omega) \) is a viscosity solution of \( H[u] = 0 \) in \( \Omega \), then

\[
u(x) = \inf_{\mathcal{A}} [d_H(x, y) + u(y) \mid y \in \mathcal{A}] \text{ for all } x \in \Omega.
\]

We remark that if \( \mathcal{A} = \emptyset \), then there exists no solution \( u \) of \( H[u] = 0 \) in \( \Omega \). The above theorem is a weaker version of [19, Theorem 10.4] which is formulated with the Mather set in place of the Aubry set \( \mathcal{A} \) and with quasi-convex Hamiltonian \( H \). In the case where \( \mathcal{A} \) is a finite set, a corresponding result for the Dirichlet problem for bounded domains has already been obtained in [24].

Another remark here is on the representation of \( d_H \) as the value function of an optimal control problem associated with the Hamiltonian \( H \). Let \( I \subset \mathbb{R} \) be an interval and \( \gamma : I \to \Omega \). We say that \( \gamma \) is a curve if it is absolutely continuous on any compact subinterval of \( I \). For \( (x, t) \in \Omega \times (0, \infty) \), let \( \mathcal{C}(x, t) \) denote the space of curves \( \gamma \) on \([0, t]\) such that \( \gamma(t) = x \).
Theorem 2.5. Let $x, y \in \Omega$. Then

$$d_H (x, y) = \inf \left\{ \int_0^t L(\gamma (s), \dot{\gamma}(s))ds \mid t > 0, \gamma \in C(x, t), \gamma(0) = y \right\},$$

where $\dot{\gamma}$ denotes the derivative of $\gamma$.

Now we turn to the case when $\Omega$ is an open bounded subset of $\mathbb{R}^n$ with regular boundary. We consider the Dirichlet problem

$$H[u] = 0 \quad \text{in } \Omega,$$

$$u|_{\partial\Omega} = g,$$

where $g$ is a given continuous function on $\partial\Omega$. For the Dirichlet problem, we have to modify the definition of the Aubry set and for this we set $\mathcal{A}_D = \mathcal{A} \cup \partial\Omega$, where $\mathcal{A}$ is defined as before. Let $g \in C(\mathcal{A}_D)$ and assume that the following compatibility condition, for the solvability of the Dirichlet problem (12) and (13), holds:

$$g(x) - g(y) \leq d_H (x, y) \quad \text{for all } x, y \in \mathcal{A}_D.$$ (14)

Theorem 2.6. Under assumption (14), the function $u \in C(\overline{\Omega})$, defined by

$$u(x) = \inf \left\{ d_H (x, y) + g(y) \mid y \in \mathcal{A}_D \right\}$$

for all $x \in \overline{\Omega}$, is a solution of (12) and (13). Moreover it is a unique solution of (12) satisfying $u|_{\mathcal{A}_D} = g$.

3. Asymptotic solutions

The following result is concerned with the unbounded domain $\Omega = \mathbb{R}^n$ and we need a further restriction on $H$. Indeed, we assume that there exist functions $\phi_i$ and $\sigma_i$, with $i = 0, 1$, such that

$$H[\phi_i] \leq -\sigma_i \quad \text{in } \mathbb{R}^n,$$

$$\lim_{|x| \to \infty} \sigma_i(x) = \infty,$$

$$\lim_{|x| \to \infty} (\phi_0 - \phi_1)(x) = \infty.$$

Also, we need the following hypothesis:

for each $x \in \Omega$, the function $H(x, \cdot)$ is strictly convex in $\mathbb{R}^n$. (15)

We introduce the spaces $\Phi_0$ and $\Psi_0$ of functions on $\mathbb{R}^n$ and on $\mathbb{R}^n \times [0, \infty)$, respectively, as

$$\Phi_0 = \{ f \in C(\mathbb{R}^n, \mathbb{R}) \mid \inf_{\mathbb{R}^n} (f - \phi_0) > -\infty \},$$
$$\Psi_0 = \{ g \in C(\mathbb{R}^n \times [0, \infty), \mathbb{R}) \mid \inf_{\mathbb{R}^n \times [0, T]} (g - \phi_0) > -\infty, \text{ for all } T > 0 \}.$$
Theorem 3.1. (a) The additive eigenvalue problem (4) has a solution \((c, v) \in \mathbb{R} \times \Phi_0\), and moreover the additive eigenvalue \(c\) is uniquely determined. That is, if \((d, w) \in \mathbb{R} \times \Phi_0\) is another solution of (4), then \(d = c\).

(b) Let \(u_0 \in \Phi_0\). Then there exists a unique solution \(u \in \Psi_0\) of the Cauchy problem (2) and (3).

(c) Let \(\mathcal{A}_c\) be the Aubry set for the Hamiltonian \(H - c\) and \(d_{H,c} := d_{H-c}\). Let \(u \in \Psi_0\) be the solution of (2) and (3). Assume that (15) holds. Then there exists a solution \(v_0 \in \Phi_0\) of (4) with \(c\) being the additive eigenvalue for \(H\) such that

\[
\lim_{t \to \infty} \sup_{x \in B(0, R)} |u(x, t) + ct - v_0(x)| = 0 \text{ for any } R > 0.
\]

Moreover,

\[
v_0(x) = \inf \{d_{H,c}(x, y) + d_{H,c}(y, z) + u_0(z) \mid z \in \mathbb{R}^n, y \in \mathcal{A}_c\} \text{ for all } x \in \mathbb{R}^n.
\]

The above result is contained in [23]. This result is a variant of those obtained by Fathi, Namah, Roquejoffre, Barles, Souganidis, Davini, Siconolfi, and others for compact manifolds \(\Omega\). We refer to [16], [26], [28], [6], [12] for previous results and developments. See also [21] for results in \(\mathbb{R}^n\) and [7], [20] for similar results for viscous Hamilton–Jacobi equations. In [21], Hamilton–Jacobi equations of the form

\[
u_t + \alpha x \cdot Du + H_0(Du) = f(x)
\]

are treated, where \(\alpha\) is a positive constant and \(H_0, f \in C(\mathbb{R}^n)\). It is assumed that \(H_0\) is convex and coercive and that there is a convex function \(l \in C(\mathbb{R}^n)\) such that

\[
\lim_{|x| \to \infty} (l(-\alpha x) + f(x)) = \infty \quad \text{and} \quad \lim_{|\xi| \to \infty} (L_0 - l)(\xi) = \infty,
\]

where \(L_0\) is the convex conjugate of \(H_0\). If we assume that \(H_0\) is strictly convex, then the hypotheses of Theorem 3.1 are satisfied with the choice of \(\phi_0(x) := -(1/\sigma)l(-\alpha x), \phi_1(x) := -(1/\alpha)L(-\alpha x), \sigma_0(x) := l(-\alpha x) + f(x) - C,\) and \(\sigma_1(x) := L(-\alpha x) + f(x),\) where \(C\) is a sufficiently large constant.

Another example of \(H\) which satisfies the hypotheses of Theorem 3.1 is given by

\[
H(x, p) = H_0(x, p) - f(x),\]

where \(H_0 \in C(\mathbb{R}^n \times \mathbb{R}^n)\) satisfies (15), (7), and

\[
\sup_{\mathbb{R}^n \times B(0, \delta)} |H_0| < \infty \quad \text{for some } \delta > 0,
\]

and \(f \in C(\mathbb{R}^n)\) satisfies \(\lim_{|x| \to \infty} f(x) = \infty\). A possible choice of \(\phi_i, i = 0, 1,\) is as follows: \(\phi_0(x) := -(\delta/2)|x|\) and \(\phi_1(x) = -\delta|x|\).

4. Outline of proof of Theorem 3.1

4.1. Additive eigenvalue problem. Additive eigenvalue problem (4) appears in ergodic optimal control or the homogenization of Hamilton–Jacobi equations. In ergodic optimal control the additive eigenvalue \(c\) corresponds to averaged long-run optimal costs while \(c\) determines the effective Hamiltonian in the homogenization of
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To avoid technicalities, we assume in this subsection that \( \phi_0 = 0 \). One method of solving problem (4) is to approximate it by a regular problem

\[
\lambda v \phi + H(x, Dv) = 0 \quad \text{in } \mathbb{R}^n,
\]

where \( \lambda \) is a positive constant, and then send \( \lambda \to 0 \) along an appropriate sequence \( \lambda_j \to 0 \), to obtain a solution \( (c, v) \in \mathbb{R} \times \Phi_1 \) of (4) as the limits

\[
c := \lim_{j \to \infty} (-\lambda_j v\phi_\lambda(0)) \quad \text{and} \quad v(x) := \lim_{j \to \infty} (v\phi_\lambda(x) - v\phi_\lambda(0)).
\]

Indeed, thanks to the coercivity of \( H \), we may build a solution \( \psi_0 \in C^{0+1}(\mathbb{R}^n) \) of \( H[\psi_0] \geq -C_0 \) in \( \mathbb{R}^n \) for some constant \( C_0 > 0 \) which satisfies \( \phi_0 \leq \psi_0 \) in \( \mathbb{R}^n \). If \( C > 0 \) is large enough, then the functions

\[
f_\lambda(x) := \phi_0(x) - \lambda^{-1} C \quad \text{and} \quad g_\lambda(x) := \psi_0(x) + \lambda^{-1} C
\]

are a subsolution and a supersolution of (16), respectively. The Perron method now yields a solution \( \phi_\lambda \in C^{0+1}(\mathbb{R}^n) \) of (16) which satisfies \( f_\lambda \leq \phi_\lambda \leq g_\lambda \) in \( \mathbb{R}^n \). Again the coercivity of \( H \) guarantees that the family \( \{v_\lambda\}_{\lambda \in (0,1)} \) is locally equi-Lipschitz in \( \mathbb{R}^n \), while the inequality \( f_\lambda \leq \phi_\lambda \leq g_\lambda \) in \( \mathbb{R}^n \) assures that \( \{\lambda v\phi_\lambda(0)\}_{\lambda \in (0,1)} \) is bounded. These observations allow us to pass to the limit in (17).

Another approach to solving (4) is to define the additive eigenvalue \( c \in \mathbb{R} \) by

\[
c = \inf \{a \in \mathbb{R} \mid \text{there exists a solution } \phi \in C(\mathbb{R}^n) \text{ of } H[\phi] \leq a \}
\]

and then to prove that \( A_c \neq \emptyset \). Any pair of \( c \) and \( v := d_{H,c}(\cdot, y) \), with \( y \in A_c \), is a solution of (4).

In what follows we assume by replacing \( H \) by \( H - c \), where \( c \) is the additive eigenvalue for \( H \), that \( c = 0 \).

4.2. Critical curves. An important tool in the weak KAM approach is the collection of critical curves for the Lagrangian \( L \). It allows us to analyze the asymptotic behavior for large time of solutions of (2) in the viewpoint of the Lagrangian dynamical system behind (2), which may not be well-defined under our regularity assumptions on \( H \).

According to [12], a curve \( \gamma : \mathbb{R} \to \mathbb{R}^n \) is said to be critical for the Lagrangian \( L \) if for any \( a, b \in \mathbb{R} \), with \( a < b \), and any subsolution \( \phi \in C(\mathbb{R}^n) \) of \( H[\phi] = 0 \) in \( \mathbb{R}^n \),

\[
\phi(\gamma(b)) - \phi(\gamma(a)) = \int_a^b L(\gamma(s), \dot{\gamma}(s))ds.
\]

We denote by \( \Gamma \) the set of all critical curves \( \gamma \). Note that, in general, if \( \gamma \) is a curve on \( [a, b] \) and \( \phi \in C(\mathbb{R}^n) \) is a subsolution of \( H[\phi] = 0 \) in \( \mathbb{R}^n \), then

\[
\phi(\gamma(b)) - \phi(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot{\gamma}(s))ds.
\]
Indeed, we compute by assuming \( \phi \in C^1(\mathbb{R}^n) \) that for any curve \( \gamma \) on \([a, b]\),
\[
\phi(\gamma(b)) - \phi(\gamma(a)) = \int_a^b D\phi(\gamma(s)) \cdot \dot{\gamma}(s) \, ds \\
\leq \int_a^b [L(\gamma, \dot{\gamma}) + H(\gamma, D\phi(\gamma(s)))] \, ds \leq \int_a^b L(\gamma, \dot{\gamma}) \, ds.
\]
Here we have used the Fenchel inequality: \( p \cdot \xi \leq H(x, p) + L(x, \xi) \) for all \( x, p, \xi \) in \( \mathbb{R}^n \). The above computation can be justified by the standard mollification technique for general \( \phi \) which is locally Lipschitz because of the coercivity of \( H \).

**Theorem 4.1.** For any \( y \in \mathcal{A} \) there exists a critical curve \( \gamma \) such that \( \gamma(0) = y \).

Existence of critical curves is one of crucial observations in weak KAM theory. See [15], [17], [18], [12] for results on the existence of critical curves.

A main point in the proof of the above theorem is the following general observation concerning the Aubry set, which gives another definition of the Aubry set in terms of the Lagrangian \( L \). We remark that this latter definition of the Aubry set has been employed in [15], [18], [19].

**Theorem 4.2.** Let \( y \in \mathbb{R}^n \). Then \( y \in \mathcal{A} \) if and only if for any \( \varepsilon > 0 \),
\[
\inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds \mid t \geq \varepsilon, \gamma \in \mathcal{C}(y, t), \gamma(0) = y \right\} = 0.
\]

Once we have the above theorem in hand, the proof of Theorem 4.1 goes like this. For any \( y \in \mathcal{A} \) and \( k \in \mathbb{N} \) we may choose a curve \( \gamma_k \) on \([0, T_k]\), where \( T_k \geq k \) such that \( \gamma_k(0) = \gamma_k(T_k) = y \) and
\[
\int_0^{T_k} L(\gamma_k(s), \dot{\gamma}_k(s)) \, ds < \frac{1}{k}.
\]
We define the curve \( \eta_k \) on \([-T_k, T_k]\) by setting
\[
\eta_k(s) = \begin{cases} 
\gamma_k(s + T_k) & \text{for } s \in [-T_k, 0], \\
\gamma_k(s) & \text{for } s \in [0, T_k].
\end{cases}
\]
Using the observations that
\[
\lim_{r \to \infty} \inf \left\{ \frac{L(x, \xi)}{|\xi|} \mid x \in B(0, R), \xi \in \mathbb{R}^n \setminus B(0, r) \right\} = \infty \quad \text{for any } R > 0, \quad (19)
\]

since \( H \in C(\mathbb{R}^n \times \mathbb{R}^n) \) and that \( \mathcal{A} \) is compact, we may send \( k \to \infty \) along a subsequence so that \( \eta_k \) has a limit \( \gamma \) in \( C(\mathbb{R}, \mathbb{R}^n) \), which is a critical curve.

For any \( \gamma \in \Gamma \) we have
\[
\gamma(t) \in \mathcal{A} \quad \text{for all } t \in \mathbb{R}. \quad (20)
\]
This can be seen easily by recalling that for any \( y \in \mathbb{R}^n \setminus \mathcal{A} \) there are functions \( \phi \in C(\mathbb{R}^n) \) and \( \sigma \in C(\mathbb{R}^n) \) such that \( \sigma \geq 0 \) in \( \mathbb{R}^n \), \( \sigma(y) > 0 \), and \( H[\phi] \leq -\sigma \) in \( \mathbb{R}^n \) and using (18), with \( H(x, p) \) replaced by \( H(x, p) + \sigma(x) \).
4.3. Cauchy problem. To prove existence of a solution of (2) and (3), we may use the well-known formula

\[ u(x, t) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s))ds + u_0(\gamma(0)) \mid \gamma \in \mathcal{C}(x, t) \right\} \quad (21) \]

for \((x, t) \in \mathbb{R}^n \times (0, \infty)\).

We have to check if this formula really gives a solution of (2) and (3). For this the first thing to do is to check that the function \(u\) defined by (21) is a locally bounded function in \(\mathbb{R}^n \times (0, \infty)\). Fix a subsolution \(\phi \in C(\mathbb{R}^n)\) of \(H[\phi] = 0\) in \(\mathbb{R}^n\) and set \(\phi_2(x) = \min\{\phi(x) - A, \phi_1(x)\}\), where \(A > 0\) is a constant. If \(A\) is large enough, then \(\phi_2\) has the following properties: (a) \(H[\phi_2] \leq 0\) in \(\mathbb{R}^n\) and (b) \(\phi_2 \leq u_0\) in \(\mathbb{R}^n\).

Then, for any \((x, t) \in \mathbb{R}^n \times (0, \infty)\) and \(\gamma \in \mathcal{C}(x, t)\), we have

\[ \phi_2(\gamma(t)) \leq u_0(\gamma(0)) + \int_0^t L(\gamma, \dot{\gamma})ds, \]

from which we get \(\phi_2(x) \leq u(x, t)\).

In order to get a local upper bound of the function \(u\) defined by (21), we fix any \((x, t) \in \mathcal{A} \times (0, \infty)\) and choose a curve \(\gamma \in \Gamma\) so that \(\gamma(t) = x\). Existence of such a critical curve is guaranteed by Theorem 4.1. As before let \(\phi \in C(\mathbb{R}^n)\) be a solution of \(H[\phi] \leq 0\) in \(\mathbb{R}^n\). We have

\[ u(x, t) \leq u_0(\gamma(0)) + \int_0^t L(\gamma, \dot{\gamma})ds \leq u_0(\gamma(0)) + \phi(\gamma(t)) - \phi(\gamma(0)) \]

\[ \leq \max_{\mathcal{A}} u_0 + 2 \max_{\mathcal{A}} |\phi|. \]

Since \(H \in C(\mathbb{R}^n \times \mathbb{R}^n)\), for each \(R > 0\) there are constants \(\delta_R > 0\) and \(C_R > 0\) such that \(L(x, \xi) \leq C_R\) for all \((x, \xi) \in B(0, R) \times B(0, \delta_R)\). Fix any \(R > 0\) such that \(\mathcal{A} \subset B(0, R)\) and any \(x \in B(0, R)\). There is a \(T_R > 0\), independent of \(x\), and a curve (e.g., the line segment connecting a point in \(\mathcal{A}\) and \(x\)) \(\gamma_s \in C^1([0, T_R], \mathbb{R}^n)\) such that \(\gamma_s(0) \in \mathcal{A}\), \(\gamma_s(T_R) = x\), and \(|\gamma_s(s)| \leq R\), \(|\dot{\gamma}_s(s)| \leq \delta_R\) for all \(s \in [0, T_R]\). Using the dynamic programming principle which states that for any \(x \in \mathbb{R}^n\) and \(t, s \in (0, \infty)\),

\[ u(x, t + s) = \inf \left\{ \int_0^s L(\gamma(\tau), \dot{\gamma}(\tau))d\tau + u(\gamma(0), t) \mid \gamma \in \mathcal{C}(x, s) \right\} \quad (22) \]

we find that

\[ u(x, t + T_R) \leq \int_0^{T_R} L(\gamma_s(s), \dot{\gamma}_s(s))ds + u(\gamma_s(0), t) \]

\[ \leq C_R T_R + \max_{\mathcal{A}} u_0 + 2 \max_{\mathcal{A}} |\phi| \quad \text{for all } t \geq 0. \]

Noting that \(u(x, t) \leq u_0(x) + t L(x, 0)\), we obtain for all \(t > 0\),

\[ u(x, t) \leq \max_{B(0,R)} (|u_0| + T_R L(\cdot, 0)), C_R T_R + \max_{\mathcal{A}} u_0 + 2 \max_{\mathcal{A}} |\phi|. \]
Setting \( u(x, 0) = u_0(x) \) for \( x \in \mathbb{R}^n \) and making further standard estimates on \( u \), we conclude the following theorem.

**Theorem 4.3.** The function \( u \) belongs to \( \Psi_0 \) and moreover \( u \) is bounded and uniformly continuous on \( B(0, R) \times [0, \infty) \) for any \( R > 0 \).

Furthermore, using the dynamic programming principle (22), we have:

**Theorem 4.4.** The function \( u \) is a solution of (2) and (3).

Uniqueness of solutions of (2) and (3) follows from the following comparison theorem.

**Theorem 4.5.** Let \( u \in C(\mathbb{R}^n \times [0, \infty)) \) and \( v \in \Psi_0 \) be a subsolution and a supersolution of (2). Assume that \( u(\cdot, 0) \leq v(\cdot, 0) \) in \( \mathbb{R}^n \). Then \( u \leq v \) in \( \mathbb{R}^n \times [0, \infty) \).

An outline of the proof of this theorem goes like this. Let \( A \) and \( B \) be large positive constants. We set \( \psi(x,t) = \phi_1(x) - At \) for \( (x, t) \in \mathbb{R}^n \times [0, \infty) \). We may fix \( A \) so that \( \psi \) is a subsolution of (2). We set \( u_B(x, t) = \min\{u(x, t), \psi(x, t) + B\} \) for \( (x, t) \in \mathbb{R}^n \times (0, \infty) \) and observe that \( u_B \) is a subsolution of (2), that \( u_B(\cdot, 0) \leq v(\cdot, 0) \) in \( \mathbb{R}^n \), and that for each \( T > 0 \),

\[
\lim_{|x| \to \infty} \sup_{[0, T]} |u_B(x,t) - v(x,t)| = -\infty. \tag{23}
\]

Applying the standard comparison result to \( u_B \) and \( v \) on the set \( B(0, R) \times [0, R) \), with \( R > 0 \) sufficiently large, we find that \( u_B \leq v \) in \( \mathbb{R}^n \times [0, R) \). Because of the arbitrariness of \( R, B \), we conclude that \( u \leq v \) in \( \mathbb{R}^n \times [0, \infty) \).

4.4. Asymptotic analysis.

4.4.1. Equilibrium points. A point \( y \) in the Aubry set \( \mathcal{A} \) is called an equilibrium point if \( \min_{p \in \mathbb{R}^n} H(y, p) = 0 \) or equivalently \( L(y, 0) = 0 \). Under the assumption that \( \mathcal{A} \) consists only of equilibrium points, the convergence assertion of Theorem 3.1 can be proved in an easy way compared to the general case. To see this, let \( u \in \Psi_0 \) be the solution of (2) and (3), and set \( v_0(x) = \liminf_{t \to \infty} u(x, t) \) for \( x \in \mathbb{R}^n \). We then observe in view of the convexity of \( H(x, p) \) in \( p \) that \( v_0 \) is a solution of \( H[v_0] = 0 \) in \( \mathbb{R}^n \). Also, we observe at least formally that \( u_t \leq 0 \) in \( \mathcal{A} \times (0, \infty) \), which can be stated correctly that the function \( t \mapsto u(x, t) \) is nonincreasing in \( (0, \infty) \) for any \( x \in \mathcal{A} \). Now, by Theorem 4.3 and Dini’s lemma, we see that the functions \( u(\cdot, t) \) converge to \( v_0 \) uniformly on \( \mathcal{A} \) as \( t \to \infty \).

We take a small digression here and state a comparison theorem for (1) with \( \Omega = \mathbb{R}^n \), a version of Theorem 2.3 for \( \Omega = \mathbb{R}^n \).

**Theorem 4.6.** Let \( u, v \in C(\mathbb{R}^n) \) be a subsolution and a supersolution of \( H = 0 \) in \( \mathbb{R}^n \), respectively. Assume that \( u \leq v \) on \( \mathcal{A} \) and that \( v \in \Phi_0 \). Then \( u \leq v \) in \( \mathbb{R}^n \).
For the proof of this theorem, we may assume by a simple modification of \( \phi_1 \) that \( \phi_1 \) is a solution of \( H[\phi_1] \leq 0 \) in \( \mathbb{R}^n \). We then replace \( u \) by \( \varepsilon \phi_1 + (1 - \varepsilon)u \), with a small \( \varepsilon \in (0, 1) \), so that we are in the situation that \( \lim_{|x| \to \infty} (u - v)(x) = -\infty \), which allows us to work on a ball \( B(0, R) \), with sufficiently large \( R > 0 \). Now, as in the proof of Theorem 2.3, we get \( u \leq v \) in \( \mathbb{R}^n \). This is an outline of the proof of Theorem 4.6.

Back to the main theme, we use the same argument as the proof of Theorem 4.6 just outlined, to control the functions \( u(\cdot, t) \) through their restrictions on \( \mathcal{A} \) and to conclude the desired convergence of \( u(\cdot, t) \) to \( v_0 \) in \( \mathbb{R}^n \) as \( t \to \infty \).

We remark that if \( \mathcal{A} \) is a finite set, then all the points of \( \mathcal{A} \) are equilibrium points. Also, if there is a function \( f \in C(\mathbb{R}^n) \) such that \( H(x, Df) \leq \min_{p \in \mathbb{R}^n} H(x, p) \) for \( x \in \mathbb{R}^n \) in the viscosity sense, then all the points of \( \mathcal{A} \) are equilibrium points.

### 4.4.2. General case.

We turn to the general case. Let \( u \in C(\mathbb{R}^n \times [0, \infty)) \) be a solution of (2). A formal calculation reveals that for any \( \gamma \in \Gamma \), any \( t, T \in [0, \infty) \) satisfying \( t < T \), and any solution of \( \phi \in C(\mathbb{R}^n) \) of \( H[\phi] \leq 0 \) in \( \mathbb{R}^n \),

\[
 u(\gamma(T), T) - u(\gamma(t), t) = \int_t^T \left[ Du(\gamma(s), s) \cdot \dot{\gamma}(s) + u_t(\gamma(s), s) \right] ds
 = \int_t^T \left[ Du(\gamma(s), s) \cdot \dot{\gamma}(s) - H(\gamma(s), Du(\gamma(s), s)) \right] ds
 \leq \int_t^T L(\gamma(s), \dot{\gamma}(s)) ds = \phi(\gamma(T)) - \phi(\gamma(t)).
\]

Indeed, we have

**Lemma 4.7.** Under the above assumptions, the function \( t \mapsto u(\gamma(t), t) - \phi(\gamma(t)) \) is nonincreasing on \([0, \infty)\).

In what follows we denote by \( S_t \), with \( t \geq 0 \), the semigroup generated by (2), i.e., the map \( S_t : \Phi_0 \to \Phi_0 \) defined by \( S_t u_0 = u(\cdot, t) \), where \( u \) is the solution in \( \Psi_0 \) of (2) and (3).

The continuous dependence of the solution of (2) and (3) on the initial data can be stated as follows.

**Theorem 4.8.** Let \( f, g \in \Phi_0 \) be a subsolution and a supersolution of \( H = 0 \) in \( \mathbb{R}^n \), respectively. Assume that \( f \leq g \) in \( \mathbb{R}^n \). Then for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( u_0, v_0 \in [f, g] \cap C(\mathbb{R}^n) \), if

\[
 \max_{B(0, \delta^{-1})} (u_0 - v_0) \leq \delta,
\]

then

\[
 \sup_{(x, t) \in B(0, \varepsilon^{-1}) \times [0, \infty)} (S_t u_0(x) - S_t v_0(x)) \leq \varepsilon.
\]
Here \([f, g]\) denotes the space of those functions \(w: \mathbb{R}^n \to \mathbb{R}\) which satisfy \(f \leq w \leq g\) in \(\mathbb{R}^n\).

This theorem can be proved by an argument similar to the proof of Theorem 4.5, and we omit presenting it here.

For any \(u_0 \in \Phi_0\), the \(\omega\)-limit set \(\omega(u_0)\) for the initial point \(u_0\) is defined as the set consisting of those \(w \in \Phi_0\) for which there exists a sequence \(\{t_j\} \subset (0, \infty)\) diverging to infinity such that \(S_{t_j} u_0 \to w\) in \(C(\mathbb{R}^n)\) as \(j \to \infty\). It is obvious from Theorem 4.3 that \(\omega(u_0) \neq \emptyset\) for all \(u_0 \in \Phi_0\). Another basic property of \(\omega\)-limit sets, which follows from Theorems 4.8 and 4.3, is the following.

**Lemma 4.9.** Let \(u_0 \in \Phi_0\) and let \(\{t_j\}, \{r_j\} \subset (0, \infty)\) be sequences diverging to infinity such that \(S_{t_j} u_0 \to v\) and \(S_{r_j + t_j} u_0 \to w\) in \(C(\mathbb{R}^n)\) as \(j \to \infty\) for some \(v, w \in \omega(u_0)\). Then \(S_{r_j} v \to w\) in \(C(\mathbb{R}^n)\) as \(j \to \infty\).

So far, we have needed only the convexity of the Hamiltonian \(H(x, p)\) in \(p\), but not its strict convexity (15) although this point may not be clear because of the rough presentation. In the next lemma we need the strict convexity assumption (15), which guarantees that \(L(x, \xi)\) and \(D_\xi L(x, \xi)\) are continuous on the set \(\{(x, \xi) \in \mathbb{R}^{2n} | L(x, \xi) < \infty\}\). The following lemma is an equivalent of \([12, \text{Lemma 5.2}]\) and a key observation for the convergence proof.

**Lemma 4.10.** Assume that (15) holds. Then there exist a \(\delta > 0\) and a function \(\rho \in C([0, \infty))\), with \(\rho(0) = 0\), such that for any \(u_0 \in \Phi_0\), \(\gamma \in \Gamma\), \(\varepsilon \in (-\delta, \delta)\), and \(t > 0\),

\[
S_t u_0(\gamma(t)) \leq u_0(\gamma(\varepsilon t)) + \int_{\varepsilon t}^t L(\gamma(s), \dot{\gamma}(s)) ds + |\varepsilon t| |\rho(|\varepsilon|)|.
\]

**Proof of Theorem 3.1 (c).** We denote by \(\omega(\Gamma)\) the set of all those curves \(\gamma\) to which there correspond a curve \(\eta \in \Gamma\) and a sequence \(\{t_j\}\) diverging to infinity such that \(\eta(\cdot + t_j) \to \gamma\) in \(C(\mathbb{R})\) as \(j \to \infty\). We set \(\mathcal{M} = \{\gamma(0) \mid \gamma \in \omega(\Gamma)\}\). We remark that any \(\gamma \in \omega(\Gamma)\) is a critical curve for \(L\). Consequently, we have \(\mathcal{M} \subset \mathcal{A}\). Moreover, it is easily seen that for any two solutions \(\phi, \psi \in C(\mathbb{R}^n)\) of \(H = 0\) in \(\mathbb{R}^n\), if \(\phi \leq \psi\) on \(\mathcal{M}\), then \(\phi \leq \psi\) on \(\mathcal{A}\).

As before we define the function \(v_0 \in \Phi_0\) by

\[
v_0(x) = \liminf_{t \to \infty} S_t u_0(x),
\]

which is a solution of \(H[v_0] = 0\) in \(\mathbb{R}^n\).

We prove that \(u(\cdot, t) \to v_0\) in \(C(\mathbb{R}^n)\) as \(t \to \infty\). To this end, it is enough to show that \(w = v_0\) for all \(w \in \omega(u_0)\). By the definition of \(v_0\), we have \(v_0 \leq w\) in \(\mathbb{R}^n\) for all \(w \in \omega(u_0)\). Hence, recalling the proof in the case when \(\mathcal{A}\) consists only of equilibrium points and using the remark made above, we find that it is enough to show that \(w \leq v_0\) in \(\mathcal{M}\) for all \(w \in \omega(u_0)\). (We omit here proving the formula for \(v_0\) in the theorem.)
Fix any \( w \in \omega(u_0) \) and \( y \in \mathcal{M} \). Choose a curve \( \gamma \in \Gamma \) and sequences \( \{a_j\} \subset (0, \infty) \) diverging to infinity so that, as \( j \to \infty \), \( \gamma(a_j) \to y \) and \( S_{b_j} w \to w \) in \( C(\mathbb{R}^n) \). Existence of such a sequence \( \{b_j\} \) is assured by Lemma 4.9. We may assume that \( c_j := a_j - b_j \to \infty \) as \( j \to \infty \). We fix any \( s \geq 0 \) and apply Lemma 4.10, with \( w \) and \( \gamma(\cdot + c_j) \) in place of \( u_0 \) and \( \gamma \), respectively, and with \( t = b_j \) and \( \varepsilon = s/b_j \), to obtain for sufficiently large \( j \),

\[
S_{b_j} w(\gamma(b_j + c_j)) - w(\gamma(s + c_j)) \leq v_0(\gamma(b_j + c_j)) - v_0(\gamma(s + c_j)) + s \rho(s/b_j),
\]

where \( \rho \in C([0, \infty)) \) is the function from Lemma 4.10. Sending \( j \to \infty \) yields

\[
w(y) - v_0(y) \leq w(\eta(s)) - v_0(\eta(s)) \quad \text{for all } s \geq 0 \tag{24}
\]

and for some \( \eta \in \omega(\Gamma) \).

The final step is to show that

\[
\liminf_{t \to \infty} [w(\eta(t)) - v_0(\eta(t))] \leq 0, \tag{25}
\]

which yields, together with (24), \( w(y) \leq v_0(y) \). To do this, we choose sequences \( \{t_j\}, \{\tau_j\} \subset (0, \infty) \) diverging to infinity so that, as \( j \to \infty \), \( S_{t_j} u_0(\eta(0)) \to v_0(\eta(0)) \) and \( S_{t_j + \tau_j} u_0 \to w \) in \( C(\mathbb{R}^n) \). We calculate by using Lemma 4.7 that

\[
w(\eta(\tau_j)) - v_0(\eta(\tau_j)) \leq |w(\eta(\tau_j)) - S_{t_j+\tau_j} u_0(\eta(\tau_j))| + S_{t_j+\tau_j} u_0(\eta(\tau_j)) - v_0(\eta(\tau_j)) \leq \max_A |w - S_{t_j+\tau_j} u_0| + S_{t_j} u_0(\eta(0)) - v_0(\eta(0)).
\]

Sending \( j \to \infty \), we get

\[
\limsup_{j \to \infty} [w(\eta(\tau_j)) - v_0(\eta(\tau_j))] \leq 0,
\]

which shows that (25) is valid. \( \square \)

References


Asymptotic solutions for large time of Hamilton–Jacobi equations


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