1. Introduction

It is a great pleasure for me to report on the recent spectacular developments concerning the Poincaré Conjecture.

Grigory Perelman has solved the Poincaré Conjecture. He has shown that, as Poincaré conjectured, any closed, simply connected 3-manifold is homeomorphic to the 3-sphere.

The paper in which Poincaré posed this problem in 1904 ([14]) marked, in my view, the founding of topology as an independent discipline within pure mathematics. Over the intervening 100 years, the problem has been much studied and generalized, and many related problems have been solved. It has been linked, in one way or another, with most of the progress in topology in the last 100 years. While related problems have been solved, the original conjecture stood untouched, resisting all attempts. Before Perelman’s work, there had been no progress on toward solving the Poincaré Conjecture, and many viewed it as the siren song of Topology, for many a boat had foundered on the rocks trying to reach it. There have been innumerable proposed proofs and proposed counter-examples, but none, before Perelman’s, withstood scrutiny.

Solving the Poincaré Conjecture is a signal achievement for Perelman, but it is also a signal achievement for all of mathematics, for it gives a measure of how far our understanding of the subject has advanced in the last 100 years. To paraphrase Newton, Perelman has seen far, but to do so he stood on the shoulders of giants who came before him. One giant, in particular, stands out. He is Richard Hamilton. Over a period of 25 years, Hamilton painstakingly built the solid and elaborate foundation upon which Perelman constructed the edifice of his proof. Without Hamilton’s work, Perelman’s would not have been possible.

One of the most interesting aspects of the resolution of the Poincaré Conjecture is the nature of the solution. While the problem is purely topological in its formulation, the proof is not. The proof uses deep techniques and results from other areas of mathematics, namely analysis and differential geometry. It is not at all clear a priori that these ideas have any relevance to the Poincaré Conjecture, but in the end they turn out to be the only way (so far) to approach this question successfully.

My goal in this article is to give you a sense of the importance, centrality, and the depth of the Poincaré Conjecture. Then I will discuss surfaces and 3-dimensional spaces and describe how topologists think about them. Next, I will formulate and
explain the statement of the conjecture. Then I will give an overview of the ideas that go into the solution – Ricci flow, Ricci flow with surgery, and finite-time extinction for Ricci flow with surgery when applied to a simply connected 3-manifold. The latter will be a very superficial view of the mathematics. I refer the reader to the body of Hamilton’s work [2] as well as to Perelman’s preprints [11], [13], and [12] for more precise statements of the analytic and geometric results that are needed.

2. Problems in mathematics

2.1. A brief history. The central role of problems in mathematics goes back to the Greeks (at least). From them we have the famous examples of the question of squaring the circle and of doubling the cube using only compass and straightedge, among others\(^1\). Starting in the 16th century mathematicians challenged each other with problems and by the 18th century learned societies posed problems to the general mathematical community often with prizes awarded for their solution. The most famous set of problems in modern mathematics is Hilbert’s 23 problems posed at the International Congress of Mathematicians in Paris in 1900; see [7]. These were posed for an entirely different reason and were of an entirely different order from most of the problems heretofore seen in mathematics. Hilbert’s goal was to lay out what he considered central problems across the entire subject, problems, whose solutions and even attempted solutions, he thought would be important, indeed central, for the development of the subject in the century that was about to begin. He introduced his problems, making it clear that he saw them being linked to the future development of the subject by saying “Who among us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during the future centuries?” Some of the problems were already well known before Hilbert’s address, for example the Riemann hypothesis, and others were formulated for the first time by Hilbert. Much progress has been made on many, but not all, of these problems. Some still remain open. And as Hilbert foresaw, they did play an instrumental role in mathematics. They formed the backdrop against which a significant portion of the mathematical development of the 20th century was measured. To solve one of Hilbert’s problems was to enter the Mathematical Hall of Fame.

A new list of seven problems was proposed in 2000 by the Clay Mathematics Foundation, the Clay Millennium Problems. As the Clay Foundation makes clear, the choice of the timing (100 years after Hilbert’s address) and the location (Paris, the same as the location of Hilbert’s address) was explicitly made to honor Hilbert’s address and the role his problems had indeed played in twentieth century mathematics. They are also hopeful that their list of problems will have a similar impact on mathematics in the twenty-first century. The problems which they introduced are called the 7 Millennium

\(^1\)For more details on the role of problems in mathematics, see the article by Jeremy Gray in [5].
Problems. Attached to each of the problems is a prize of $1,000,000 for its successful solution. There is one problem on this list in common with Hilbert’s list, the Riemann hypothesis. The other six are more recent problems.

One of the Clay Millennium Problems is the Poincaré Conjecture. Aside from the Riemann Hypothesis it is the oldest on the list, dating from 1904. It is the problem whose solution we are celebrating. It is the first of the Millennium Problems to be solved. None of the others seems ripe for solution. But of course, before its solution neither did the Poincaré Conjecture.

2.2. The role of problems. What makes a good problem and what role do problems play in mathematics? Quoting again from Hilbert’s address to the International Congress in 1900, “I should still more demand for a mathematical problem if it is to be perfect; for what is clear and easily comprehended attracts, the complicated repels us. Moreover, a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts. It should be to us a signpost on the tortuous paths to hidden truths, ultimately rewarding us by the pleasure in the successful solution.”

Hilbert is making several points here. If a problem is sufficiently difficult that it cannot be immediately solved but if it is not so difficult as to be inaccessible, then it will stimulate much mathematical activity and progress as different approaches are tried. History is replete with examples of extraordinary mathematics being created in an attempt to solve a long-standing problem. Sometimes these attempts result in partial solutions or further clarifications of the problem under consideration, other times, the mathematics does not reach its intended target, but ends up being useful in a completely different area. Good problems stimulate mathematical activity both directly related to the problem and in other surrounding areas of mathematics.

Another point that Hilbert is making is that problems become more famous as they resist more and more different attempts at solution, and that, when they reach this status, they are used a measuring stick against which the power of new ideas is tested. If a new idea makes progress on an old and famous problem, then it has demonstrated its originality, depth and power. Of course, solving such long-standing problems bestows honor in the first instance on the solver because he or she has succeeded where all others failed, but also most of the time it is a marker for the progress of the discipline as a whole. Mathematics has matured to the point where rarely, if ever, is the solution of a significant problem the result entirely of the work of a single mathematician. Rather such advances rest on the general advancement of understanding of the subject and the various previously developed techniques available to attack the given problem. In the case of the Poincaré Conjecture I have already referred to the indispensable work of Hamilton on Ricci flow, but Perelman’s argument also rests on the modern theory of Riemannian manifolds and the modern theory of various compactness results for spaces of Riemannian manifolds and more general singular objects. This theory has been developed over the last 50 to 60 years by an entire army of mathematicians. Hamilton’s work in turn relies on the progress
in partial differential equations on manifolds, especially parabolic equations such as the heat equation and the mean curvature flow equation. Again the workers who developed these techniques are too numerous to list.

To me, the most amazing thing about mathematics is that there are mathematical problems that are hard enough that their solution requires decades if not centuries of work, and yet it is possible by dint of long hard work, many ideas, and incremental advances over time to arrive at a solution. Once the perspective is correct and the technical power is sufficiently developed, they succumb. They are hard not because they are computationally difficult, but because they are conceptually difficult; yet they are not conceptually too difficult that human beings are incapable of solving them. It just takes us a while to get the perspective correct, to get the right position and with the right frame of mind to solve them. That the human race is capable of such advances is cause for celebration by all of us.

3. The Poincaré Conjecture

This brings us to the problem whose solution we are discussing – the Poincaré Conjecture. This problem was originally formulated in 1904 by Henri Poincaré [14] near the end of a long article on 3-dimensional topology. This article laid out many of the basic tenets of topology and marked its beginning as an independent field of mathematical study. At the end of the article, Poincaré states that there remains one central question to be addressed, “Is every simply connected 3-manifold topologically equivalent to the 3-sphere?” In an interesting twist of history, this is not Poincaré’s first formulation of such a question. Several years earlier he had asked a similar question where the hypothesis of simply connected is replaced by a related (but we now know) weaker condition. After posing this question, Poincaré realized that he knew how to construct counter-examples to that question and that the way to show that they were indeed counter-examples was to use a topological invariant that he had invented about 10 years before; we call it the fundamental group, and in French it is called the ‘groupe de Poincaré.’ Anyway, using this invariant he showed the answer to his first question was ‘no’ and then he formulated the question that lasted 100 years.

The Poincaré Conjecture has all the attributes of an excellent problem. It was simple (for the mathematician) to state. It was a generalization of a well-known property of surfaces, so it was a natural guess as to a fundamental fact about 3-dimensional manifolds. While simple to state and while being an obvious generalization of a known mathematical result, it was not easy to resolve, or indeed to make any progress at all on this problem. The problem is a problem purely in topology: the hypotheses are topological and the conclusion is topological. It was attacked by direct topological means for 100 years without any progress; see [18]. Nevertheless, all this effort was not futile. We learned much, just not about this question.

\[\text{footnote}{2} \text{For a formal definition of this group and all other technical terms see the appendix.}\]
3.1. Mathematics generated by the Poincaré Conjecture. The Poincaré Conjecture is an attempt to characterize topologically the simplest of all 3-dimensional manifolds, the 3-sphere. In approaching this question, many techniques were developed to study 3-dimensional manifolds. Some of the most important go back to Papakyriakopoulos [10] in the 1950s. Among these are Dehn's lemma and the loop theorem and the sphere theorem. These are incredibly powerful tools for studying 3-manifolds. For example, they allow one to prove the analogue of the Poincaré Conjecture for knots in the 3-sphere. More precisely, Parakyriakipoulos showed that a knot in the 3-sphere is topologically equivalent to the unknotted circle if and only if the fundamental group of the space obtained by removing the knot from the 3-sphere is the same as the fundamental group of the space obtained by removing the trivial unknotted circle from the 3-sphere. Using these techniques Waldhausen [20], in the 1960s, gave a complete characterization of a large class of 3-manifolds, but unfortunately from the perspective of the Poincaré Conjecture, the class that Waldhausen characterized is at the other end of the spectrum from simply connected 3-manifolds. Here, we see attempts to solve the Poincaré Conjecture leading to enormous progress in a closely related area – the study of other 3-manifolds – but saying nothing about the original conjecture. But this is just the beginning of the story.

In 1960 Smale [17], in one of the most revolutionary advances in topology, realized that it was not the case that manifolds were harder and harder to study as their dimension increased. Before Smale the thinking was: surfaces are understood; we cannot prove central results about 3-dimensional manifolds, so those of higher dimension must be even harder, too hard to even begin to think about until we understand 3-dimensional manifolds. Smale generalized the Poincaré Conjecture to all dimensions (this was fairly obvious) and then proceeded to solve it in the affirmative in all dimensions 5 and higher. This was the revolution. Four years earlier Milnor [8] had shown that a closely related question was false starting in dimension 7. In particular, Milnor showed that the smooth version3 of the Poincaré Conjecture was not true in higher dimensions. Smale and Milnor each won Fields Medals for the works we have just cited. But this was just the beginning of a 20 year period of unparalleled advances in topology. Using the ideas of Smale and Milnor and others such as René Thom, topologists succeeded in answering almost every question that could possibly be answered about manifolds of dimension 5 and greater. This whole area of topology is known as ‘surgery theory,’ or ‘Browder–Novikov surgery theory’ after its two main developers; see [1]. The reason that high dimensional manifolds are easier to study than the ones of dimensions 3 and 4 is that in them there is enough room to move submanifolds (e.g., loops and surfaces) around and put them in good position with respect to each other, whereas in lower dimensional manifolds this is not possible.

By the late 1970s high dimensional manifolds were well understood, and the attention of topologists reverted back to their ‘problem children’ – dimensions 3 and 4 – and in particular to the Poincaré Conjecture. It was not clear how to proceed. Flushed

3See the appendix.
with the success in high dimensions (and dare we say the hubris it engendered), many topologists, myself included, felt that it was just a matter of time before the ‘so-called’ low dimensions would succumb to our purely topological techniques. Others, in particular Shing-Tung Yau, argued strongly that one needed geometric and analytic tools, for example minimal surfaces and special metrics, to attack these low dimensions. From a different direction, Atiyah and Singer were saying ‘Physics takes place in the low dimensions and could well make an impact.’ The history of the low dimensions is still being written but the verdict is in and is clear: topologists like myself were wrong; Yau was right; and Atiyah and Singer were right. The evolution to using ever more analysis, geometry, and physics to attack questions about manifolds of dimensions 3 and 4 is the next part of the story of topology and the Poincaré Conjecture.

3.2. **Thurston’s generalization.** The next advance in 3-dimensional topology dates from the late 1970s and early 1980s. Thurston was applying geometric ideas, in particular hyperbolic geometry\(^4\), to 3-dimensional manifolds. His work led him to formulate a general conjecture about all 3-dimensional manifolds, a conjecture that says that they could be cut apart in a very precise way into pieces that had homogeneous 3-dimensional geometries; see [19] and [15], and see also the appendix. The list of all possible 3-dimensional homogeneous geometries was known classically. Examining this list one sees that by far the most interesting case is the hyperbolic geometry that Thurston had been studying. It is also immediate from studying the list of possibilities that the Poincaré Conjecture is a very special case of Thurston’s more general conjecture. So here we have the first real progress on the Poincaré Conjecture. This progress consists in embedding the Poincaré Conjecture as a special case of a vastly more general conjecture about all 3-dimensional manifolds. Thurston’s work also established many special cases of his general conjecture, but not a case that related directly to the Poincaré Conjecture. One effect of this was that Thurston’s work convinced most topologists that his conjecture and therefore the Poincaré Conjecture was most likely true. (Some would have said definitely true.) For his work Thurston received a Fields Medal in 1982.

3.3. **Resolution in dimension four.** In the early 1980s there were two advances in dimension four. M. Freedman [4] managed to push the higher dimensional arguments down into dimension four and to prove the four-dimensional version of the Poincaré Conjecture, leaving only the original 3-dimensional version of the conjecture open. This argument required crinkling various surfaces inside the four-dimensional manifold infinitely badly in order to get them to fit and the argument could not be made to work with smooth surfaces. Freedman’s work not only solved the four-dimensional Poincaré Conjecture, it applied to all simply connected four-manifolds. (Unlike dimension three, there are many simply connected four-manifolds.) At almost the same time, Donaldson [3], using the Yang–Mill’s equations from physics, showed that

\(^4\)See appendix.
the analogues of Freedman’s results definitely did not hold for smooth manifolds. Freedman and Donaldson each received a Fields Medal in 1986 for their work on four-dimensional manifolds.

At this point, 1986, the situation is the following: The Generalized Poincaré Conjecture has been resolved affirmatively in all dimensions except dimension three. It is understood that in dimensions four and higher there is a difference between studying smooth manifolds and topological manifolds, something that was unsuspected by Poincaré and anyway does not occur in dimension three. Manifolds of dimension five and higher, and simply connected topological 4-manifolds were well understood. For all the work in topology related to the Poincaré Conjecture a total of five Fields Medals over a period of 28 years had been awarded. At the end of this unbelievable fertile period the main outstanding problem was exactly the same as it was at the beginning – the Poincaré Conjecture in its original formulation. Furthermore, all direct topological attacks on this problem had yielded no results at all – no special cases had been solved, no reductions of the problem had been made that showed promise of yielding essential new insights. While there had been incredible advances in understanding manifolds, what had been clear from the 1950s had been confirmed by these advances: the Poincaré Conjecture was the central problem in topology.

4. Method of solution

There was, however, progress being made in a different area of mathematics that would eventually pave the way for a solution of the Poincaré Conjecture. In 1982 Richard Hamilton had developed enough of the theory of the Ricci flow to prove that a compact 3-manifold admitting a Riemannian metric\(^5\) of non-negative Ricci curvature in fact admits a Riemannian metric of constant positive sectional curvature; see [6]. In particular, if such a manifold is simply connected, then a classical theorem, essentially going back to Riemann, implies that the Riemannian manifold is isometric to the 3-spheres. In particular, the manifold is diffeomorphic to the \(S^3\). While this might seem significant progress on toward the Poincaré Conjecture, the fact that the hypothesis of Hamilton’s theorem is geometric (non-negative Ricci curvature) and the fact that there was no known way to construct such metrics, meant that while this was a beautiful theorem in geometric analysis it was not apparent that it represented real progress on the Poincaré Conjecture. It suggests that it might be possible to attack the Poincaré Conjecture in this way, but whether this method is fruitful for such an attack had to await further developments. For a general survey of Ricci flow, including most of Hamilton’s papers, see [2].

There was, however, an interesting relationship, which according to Hamilton was first pointed out to him by Yau, between Thurston’s more general conjecture and Ricci

\(^5\)We discuss Riemannian metrics and curvature in more detail in Section 6 and Ricci flow in more detail in Section 7.
flow. This relationship operates at two levels. To a first approximation, Thurston’s conjecture posits that a 3-manifold admits a nice metric. Thus, in this more general conjecture the hypotheses remain topological but the conclusion is geometric. If we are searching for such a nice metric, then we can hope to find it by geometric or analytic techniques. Based on the analogy with the heat flow, one expects Ricci flow to produce such nice metrics. Indeed, in Hamilton’s result, cited above, this is exactly what happens. This is the reasoning that led Hamilton to hope to apply his evolution equation for a Riemannian metric, the Ricci flow equation, to the more general problem of constructing homogeneous metrics on 3-manifolds, that is to say to attack Thurston’s more general conjecture by using Ricci flow. But this reasoning can be pushed further to operate at a deeper level. Thurston’s conjecture requires a cutting or a surgery of 3-manifolds before finding pieces admitting nice metrics. On the other hand, the Ricci flow equation, like most non-linear parabolic equations, develops finite-time singularities that must be dealt with. It seems conceivable that these issues are of a similar nature. Maybe cutting away the finite-time singularities in order to continue the Ricci flow would exactly lead to the cutting process required in Thurston’s conjecture. This then was the idea: use the Ricci flow on all 3-manifolds and the singularity development will exactly mimic the cutting process required by Thurston’s conjecture.

To study the Ricci flow and to prove results that are relevant to 3-dimensional topology requires a detailed and delicate analytic and geometric analysis of the Ricci flow equation and the properties of its solutions. So approaching the Poincaré Conjecture and the more general geometrization conjecture of Thurston’s in this way changes the mathematical techniques and results required to solve the topological problem from topological ones (which had been attempted without success) to geometric and analytical ones, which are more refined and hence hold out the possibility of being more powerful, but of course use many deep analytic results and delicate analytic techniques.

Hamilton established some beautiful results about singularity development in Ricci flow on 3-manifolds which were suggestive that the entire program might be made to work, but he could not get good enough control on the singularities that develop in finite time to prove that he could always do surgery and continue the process. Here is where Perelman enters the story. He realized that in addition to the properties that Hamilton had established there was one more crucial one that Hamilton had not considered, a volume non-collapsing result. Perelman introduced a beautiful new concept, unlike anything in Hamilton’s work, that allowed him to establish that this extra property holds in general when dealing with Ricci flow on compact 3-manifolds. Then using a delicate combination of blow-up limits and inductive arguments he was able to show that one could always do surgery, and that after surgery the same properties hold. This allowed him to repeatedly perform surgery and create a more general flow, which is called a Ricci flow with surgery, defined for all positive time. It remained only to show that the limits as time goes to infinity of a Ricci flow with surgery satisfies Thurston’s conjecture to conclude that the initial manifold does. In the case
of the simply connected manifolds, Perelman showed that the Ricci flow with surgery eventually leads to an empty manifold. (This is analogous to, though more complicated than, Hamilton's result about what happens when one starts with positive Ricci curvature.) From this Perelman immediately deduced the Poincaré Conjecture.

5. Statement of the Poincaré Conjecture

Poincaré derived his conjecture for 3-manifolds by arguing by analogy from what was well known for surfaces by 1900. Recall the classification of surfaces. (All surfaces are implicitly closed, and oriented.) These are classified by one invariant: the genus $g$, which is an integer $\geq 0$. The two-sphere is the only surface, up to equivalence, of genus 0. The torus, i.e., the surface of a doughnut, is the only surface of genus 1, etc. One can view the surface of genus $g$ as the result of removing from a two-sphere $g$ pairs of disks (with all $2g$ disks being disjoint) and sewing in a cylinder (i.e., an annulus) between each pair of boundary circles, so that $g$ annuli are glued to the sphere with the disks removed.

Let me say a word about what a surface is and what equivalence means in this context. A surface is a space that locally looks like the Euclidean plane. This means that near every point one can impose two local coordinate functions that behave like the $x$ and $y$ coordinates in the plane near some point. For example, on the surface of the earth in a Mercator projection we normally use latitude and longitude. Of course, near the pole we use polar coordinates. On a torus we could use the angles in each of the product circles. There is no requirement here that the coordinates extend over the entire surface or even almost all of it, they need only be defined in a neighborhood of a point. But each point has such local coordinates. There is also no assumption about how different systems of coordinates are related to each other. What I am describing here is technically a topological manifold. A closely related notion is that of a smooth of $C^\infty$-manifold. Here as one passes from one coordinate system to another there is an assumption that the coordinate functions in one system are smooth, i.e., infinitely differentiable, functions when expressed in terms of the other coordinate system. It is important to note that there are not chosen or distinguished coordinates near any point. All possible coordinate systems are on an equal footing. Thus, there is no natural notion of a metric or distance function on a topological or smooth surface. The advantage of smooth manifolds is that one can do calculus, have differential equations, etc. So, for example, a Riemannian metric only makes sense on a smooth manifold, so that the Ricci flow equation exists for smooth manifolds (with Riemannian metrics) but not for topological manifolds.

Now to the notion of equivalence. For topological manifolds it is homeomorphism. Namely, two topological manifolds are equivalent (and hence considered the same object for the purposes of classification) if there is a homeomorphism, i.e., a continuous bijection with a continuous inverse, between them. Two smooth manifolds are equivalent if there is a diffeomorphism, i.e., a continuous bijection with a
continuous inverse with the property that both the map and its inverse are smooth maps, between the manifolds. Milnor’s examples were of smooth 7-dimensional manifolds that were topologically equivalent but not smoothly equivalent to the 7-sphere. That is to say, the manifolds were homeomorphic but not diffeomorphic to the 7-sphere. Fortunately, these delicate issues need not concern us here, since in dimensions two and three every topological manifold comes from a smooth manifold and if two smooth manifolds are homeomorphic then they are diffeomorphic. Thus, in studying 3-manifolds, we can pass easily between two notions. Since we shall be doing analysis, we work exclusively with smooth manifolds.

The statement that there is a unique smooth surface up to equivalence for each $g \geq 0$ means that associated to every (closed, oriented) smooth surface is an invariant, called the genus (it is the number of holes), and for every $g \geq 0$ there is a smooth surface of genus $g$ and any two such are diffeomorphic. Above, I have briefly described how to construct a surface of genus $g$ for any $g \geq 0$. Now to the punch line for surfaces, the jumping-off point for Poincaré. For any surface of genus $g > 0$ (i.e., for any surface not equivalent to the sphere) there is a loop on the surface that cannot be continuously deformed to a point; namely, take a loop that ‘goes around’ one of the holes. The situation for the two-sphere is the opposite. Any loop on the two-sphere can be continuously deformed to a point: Imagine that the loop misses the north pole and then contract it along lines of longitude to the south point.

Thus, the simplest of all surfaces, the 2-sphere, is characterized by the property that every loop on the surface deforms continuously to a point. That is to say, the sphere is, up to equivalence, the only surface with this property. If every loop in a space deforms continuously to a point, then we say that the space is simply connected.

Poincaré Conjecture is the conjecture that the analogous statement holds for (closed, oriented) 3-manifolds.

**The Poincaré Conjecture.** A (closed) 3-manifold is topologically equivalent to the 3-sphere if and only if it is simply connected.

The argument showing that the 2-sphere is simply connected applies equally well to the 3-sphere, or indeed any sphere of any dimension greater than 1. Thus, the real import of the Poincaré Conjecture is that a simply connected 3-manifold is topologically equivalent to the 3-sphere. For more details on the history of the Poincaré Conjecture see [9].

**5.1. A description of the 3-sphere.** How should we think of the 3-sphere? By definition, it is the subset of points in Euclidean four-space at distance one from the origin:

$$S^3 = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \}.$$

Stereographic projection from the north pole $(0, 0, 0, 1)$ gives an identification of the complement of the north pole in $S^3$ with the Euclidean 3-space, so that we can view $S^3$ as a compactification of $\mathbb{R}^3$ by adding one point at infinity. More useful for
what follows, we can identity each of the northern hemisphere $S^3 \cap \{ x_4 \geq 0 \}$ and the southern hemisphere $S^3 \cap \{ x_4 \leq 0 \}$ with 3-balls, and then realize $S^3$ as the union of two 3-balls with their boundaries glued together (or identified with each other).

5.2. Presentation of any 3-manifold. This latter description of the 3-sphere has a generalization that can be used to present every 3-manifold, up to equivalence. This presentation uses solid handlebodies. Consider a compact 3-manifold with boundary obtained in the following way. Begin with the compact 3-ball in 3-space and attach some number, $g$, of solid handles (two-disks cross the interval) along their ends (two-disks cross the boundary of the interval). This makes a solid subset of 3-space whose boundary is a surface of genus $g$. The solid (i.e., 3-dimensional object) is called a solid handlebody of genus $g$. We can make a 3-manifold by taking two solid handlebodies of genus $g$ and gluing their entire boundaries together by a topological equivalence. A note of caution is probably in order here: there are many ways to do this gluing, i.e., many essentially different topological equivalences between the boundaries. Using different equivalences to glue will often result in different 3-manifolds. It is a fairly direct theorem in topology that every 3-manifold is obtained by this construction for some $g$ and some gluing identification. Such a presentation of a 3-manifold is called a Heegaard decomposition and the genus of the handlebodies is called the genus of the Heegaard decomposition. Gluing two 3-balls together by an equivalence of the $S^2$ always yields a 3-manifold equivalent to $S^3$. Thus, the 3-sphere is characterized as the only 3-manifold with a Heegaard decomposition of genus 0. It is also a direct computation to deduce from the Heegaard decomposition of a 3-manifold a presentation for the fundamental group of the resulting 3-manifold. Of course, the problem here is that the 3-sphere has many other Heegaard decompositions. In fact every 3-manifold has infinitely many different Heegaard decompositions, and indeed has Heegaard decompositions of arbitrarily high genus. The direct topological approach to proving the Poincaré Conjecture is to use the information that the manifold is simply connected, which gives some information about the Heegaard decomposition, and use this information to show that the Heegaard decomposition can be reduced by the allowable moves to one of genus zero. Fortunately, there is a finite (and quite short) list of ‘moves’ that describe how to get from one Heegaard decomposition to any other. Unfortunately, this process is ineffective in the sense that knowing, say, the genus of the two-decompositions there is no bound to the number of moves that may be required nor the maximal genus of the Heegaard decompositions that occur in the ‘path’ of moves connecting the two given ones.

Another possibility, similar in spirit to much of the recent work in low dimensional topology, would be to establish a counter-example to the Poincaré Conjecture by defining a new invariant associated to Heegaard decompositions that remained invariant under the allowable moves, an invariant beyond the fundamental group, and then find a Heegaard decomposition that was simply connected yet where this more refined invariant differed from that of the 3-sphere.
In spite of much effort over a period of 100 years, no one was ever able to carry out either of these approaches successfully.

6. Riemannian metrics and Curvature

There was no progress on the Poincaré Conjecture coming from a direct topological attack. Rather the mathematical progress that would eventually lead to the solution was coming from the study of a certain evolution equation, the Ricci flow equation, for Riemannian metrics on manifolds. In order to set the stage for describing these advances, we leave the realm of topology and pass to differential geometry, in particular Riemannian geometry. A Riemannian metric on a smooth manifold is a smoothly varying, positive definite inner product on the tangent spaces of the manifolds. These inner products allow us to measure lengths of tangent vectors and angles between tangent vectors at the same point of the manifold. To say that the inner products vary smoothly means that the inner product of two smooth vector fields is a smooth function. Once we have a Riemannian metric, we can measure the length of any smooth curve in the manifold, and then by minimizing the lengths of smooth curves with given endpoints we construct an ordinary distance function (i.e., metric) on the manifold. The Riemannian metric however is more subtle and powerful than the resulting distance function. A diffeomorphism between Riemannian manifolds is said to be an isometry if it preserves the Riemannian metrics and the manifolds are said to be isometric. For example, the 3-sphere receives a Riemannian metric from its natural embedding in Euclidean 4-space. Given two tangent vectors to the 3-sphere, their inner product is their usual inner product in 4-space.

Let us try to understand the nature of the space of all Riemannian metrics on a given manifold. Associated to any manifold there is an infinite dimensional vector space of all smoothly varying contravariant, symmetric two-tensors on that manifold. Inside this vector space is the open cone of positive definitive ones. This positive cone is the space of Riemannian metrics on the manifold. If we work in a set of local coordinates \((x^1, \ldots, x^n)\) on the manifold, then the metric is written as

\[ g_{ij}(x^1, \ldots, x^n)dx^i \otimes dx^j \]

where \(g_{ij}\) is a symmetric matrix of smooth functions on the coordinate patch, positive definite at every point of the coordinate patch. Of course, if we change the local coordinates the matrix \(g_{ij}\) changes; it transforms as a tensor. This means that one cannot view the matrix itself as an invariant of the metric since it depends on the local coordinates we choose to express the metric. One can ask for example, for which (local) metrics are there local coordinates in which the metric is the usual Euclidean metric \(g_{ij}(x^1, \ldots, x^n) = \delta_{ij}\)? The answer goes back to Riemann, and we will attempt to explain it below. We shall also need the inverse to the metric: in the local coordinates it is denoted \(g^{ij}\), where \(g^{ij}\) is the inverse matrix to \(g_{ij}\).
Let us begin with the case of surfaces where the results essentially go back to Gauss. Let $\Sigma$ be a surface with a Riemannian metric and consider a point $p \in \Sigma$. For each $r > 0$ sufficiently small, the ball $B(p, r)$ of radius $r$ centered at $p$ will have an area $A_{\Sigma, r}(r)$. If $\Sigma$ is the Euclidean plane then $A_{\Sigma, r}(r) = \pi r^2$. Gauss curvature is a measure of the difference of the area $A_{\Sigma, r}(r)$ and $\pi r^2$. More precisely, we consider

$$K_{\Sigma}(p) = \lim_{r \to 0} \frac{12(\pi r^2 - A_{\Sigma, r}(r))}{\pi r^4}.$$ 

It turns out that this limit exists and is finite, and the result is a smooth function on $\Sigma$. This function is called the Gauss curvature of $\Sigma$. The intuitive idea is the following. If we take the cap of an orange peel, then there is less area in this cap than in a disk in the plane of the same radius. This is evident if we press the orange peel flat. It will tear because there is not enough of it to be pushed flat. This deficit of the area as compared to Euclidean area is a reflection of the fact that the orange peel (which is basically part of a 2-sphere) has positive curvature. On the other hand, if we perform the same construction with a small disk on a rolled up cylinder, then it will flatten out without tearing since in fact we could have made the cylinder in the first place from rolling up a flat sheet of paper and this rolling does not change the Riemannian metric. The cylinder is flat, that is to say it has zero Gauss curvature.

For higher dimensional manifolds curvature is a much more complicated algebraic object than a smooth function on the manifold. Every two-dimensional direction at every point in the manifold has a Gauss-type curvature, called the sectional curvature in that direction at that point. These fit together to make a contravariant four-tensor called the Riemannian curvature tensor. The best way to view it is the following: to each two-plane in the tangent space at a point we have a sectional curvature, which is a number. These sectional curvatures fit together to define a quadratic form, the Riemann curvature tensor, on the linear space generated by the two-planes at each point. In terms of local coordinates the Riemann curvature tensor is expressed as

$$\text{Rm} = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l.$$ 

It has three symmetry properties: (i) skew symmetry in the first two variables, (ii) skew symmetry in the last two-variables, and (iii) symmetry under interchange of the first pair and the second pair. The first property is expressed by $R_{ijkl} = -R_{jikl}$. It is a direct consequence of the definition. The second property is expressed by $R_{ijkl} = -R_{jikl}$. It is a consequence of the fact that $R_{ij}$ is an infinitesimal orthogonal automorphism of the tangent spaces, and the usual fact that the Lie algebra of the orthogonal group is the Lie algebra of skew symmetric matrices. The third property is expressed by $R_{klij} = R_{ijkl}$.

We will often denote the Riemann curvature tensor of a Riemannian manifold at a point $x$ by $\text{Rm}(x)$. In the case of a flow of metrics $g(t)$ we will denote the Riemann curvature tensor at the point $x$ under the metric $g(t)$ by $\text{Rm}(x, t)$.

According to results that go back to Riemann, the basic invariant of a Riemannian metric is its curvature, that is to say, its Riemann curvature tensor. For example, there
are local coordinates in a point of a Riemannian manifold in which the metric becomes the usual Euclidean metric if and only if the Riemann curvature tensor vanishes near the point in question. Similarly, a neighborhood of a point \( p \) in a Riemannian manifold is isometric to an open subset in the sphere of radius \( r \) if and only if all the sectional curvatures are constant and equal to \( r^{-2} \) near that point (or equivalently, if the Riemann curvature tensor viewed as a symmetric endomorphism of the second exterior power of the tangent bundle is diagonal with diagonal entries \( r^{-2} \)).

In the end, it is this result that is used to finish off the proof of the Poincaré Conjecture. Every smooth manifold has a Riemannian metric; but unfortunately it has an infinite dimensional space of them. Nevertheless, we can view the Poincaré Conjecture as saying that any simply connected manifold has a metric of constant positive curvature. The way to find this metric is to use Ricci flow starting with any Riemannian metric and show that under this evolution the metric tends to a Riemannian metric of constant sectional curvature. From this it is an easy and classical step to show that the Riemannian manifold is isometric to the 3-sphere.

7. Ricci curvature and the Ricci flow equation

A Riemannian metric \( g \) on a manifold is a point in an open cone in the infinite dimensional vector space of sections of the symmetric square of the cotangent bundle of the manifold. It follows that, formally at least, the tangent space to the space of all metrics on a manifold is the vector space of sections of the symmetric square of the cotangent bundle of the manifold. The Riemann curvature is a section of the symmetric square of the second exterior power of the cotangent bundle and hence does not have the correct tensor structure to be a tangent vector to the space of Riemannian metrics. There is however a curvature derived from the Riemann curvature tensor that does have the correct tensor structure. That is the Ricci curvature. It is the symmetric two-tensor given in local coordinates by

\[
\text{Ric}(g) = \frac{\text{Ric}}{g^{ij} R_{ijkl}} dx^i \otimes dx^k
\]

where

\[
\text{Ric}_{ij} = \text{Ric}^{ij} R_{ijkl}
\]

is the trace of the Riemannian curvature tensor on the second and fourth indices. The symmetry properties of the Riemann curvature tensor translate into the fact that \( \text{Ric}_{ij} = \text{Ric}_{ji} \), i.e., that the Ricci curvature is a symmetric two-tensor. The Ricci flow equation as written down by Hamilton is

\[
\frac{\partial g}{\partial t} = -2 \text{Ric}_g,
\]

or written in local coordinates

\[
\frac{\partial g_{ij}}{\partial t} = -2 \text{Ric}_{ij}.
\]

A solution, or a \textit{Ricci flow}, is a smooth one-parameter family of metrics \( g(t) \), parameterized by \( t \) in a non-degenerate interval, on a fixed smooth manifold, a family
satisfying this equation. Again we use the notation $\text{Ric}(x)$ to denote the Ricci curvature at the point $x$ and for a Ricci flow we denote by $\text{Ric}(x, t)$ the Ricci curvature at the point $x$ under the Riemannian metric $g(t)$.

The Ricci flow equation is an evolution equation for the Riemannian metric on a manifold, modeled on the heat equation which is mathematical model for heat flow. The intuition is that this equation should distribute the curvature equally over the manifold in much the same way as the heat equation distributes heat equally. There is one significant difference between these two situations, a difference that arises because the Ricci flow equation is non-linear. It has a correction term that is quadratic in the curvature (but involves no derivatives of the curvature). This quadratic term becomes dominant in regions where the curvature is large. As a consequence, this evolution equation can (and often does) develop singularities in finite-time. As the singularities develop the curvature is becoming unbounded and the non-linearities govern the equation.

There is one other curvature that plays an important role in the story, that is the scalar curvature. It is the trace of the Ricci curvature: $R = g^{ik} \text{Ric}_{ik}$. The scalar curvature at a point $x$ is denoted $R(x)$ and in the case of a Ricci flow the scalar curvature at a point $x$ under the metric $g(t)$ is denoted $R(x, t)$.

### 8. Applying Ricci flow to find good metrics

Hamilton showed that given a compact Riemannian manifold then there is a solution to the Ricci flow equation with this manifold as the initial condition and this solution is unique. Of course, one issue that plays a huge role when one uses this evolution equation is the development of singularities at finite-time $T < \infty$. As we indicated above these occur when the norm of the Riemann curvature (or indeed the scalar curvature) of $(M, g(t))$ becomes unbounded as $t$ approaches $T$. Let me give one simple example of this phenomenon. Begin with an $n$-sphere $(S^n, g_0)$ of constant sectional curvature $(n - 1)$, then the Ricci curvature is equal to the metric: $\text{Ric}(g_0) = g_0$. We see that $g(t) = (1 - 2t)g_0$ solves the Ricci flow equation. Notice that this solution becomes singular at $T = 1/2$, and that as $t$ approaches $T$, the manifold $(M, g(t))$ shrinks to a point in the sense that its diameter goes to zero. Also, notice that its sectional curvatures go uniformly to infinity as we approach the singular time $t = 1/2$.

Another closely related example is to take as the initial manifold $S^2 \times \mathbb{R}$ where the Riemannian metric is the product of a round metric on $S^2$ with the usual Euclidean metric on $\mathbb{R}$. Then the Ricci flow is the product of a shrinking family of round 2-spheres with the trivial flow on $\mathbb{R}$. Again we have a finite-time singularity and as we approach the singularity the manifold is shrinking to a line.

There is a more general result along these lines due to Hamilton. Suppose that $(M, g(0))$ is any compact Riemannian 3-manifold of positive Ricci curvature. Then the Ricci flow $(M, g(t))$ develops a finite-time singularity at some time $T < \infty$ and as $t \to T$ from below, the manifolds $(M, g(t))$ are shrinking to points and the metric
is becoming round. This example is reassuring in the sense that even though a finite-
time singularity develops, as that singularity develops, the metric (rescaled to have a
constant diameter) is converging to the metric for which we are searching.

In one way these examples are prototypical; in another they are very special. What
is typical is that in general every finite-time singularity is associated with manifolds
of non-negative curvature. What is atypical is that, unlike these examples, in general
singularities develop along proper subsets of the manifold not everywhere in the
manifold. This means that in order to prove results about the topology or geometry of
the entire manifold one must extend the Ricci flow past the finite-time singularities.
This requires a detailed and refined understanding of models for all singularities that
can arise at finite time. It also requires extending all the basic results about Ricci flows
to the more general flows that are constructed in extending past the singularities.

Hamilton laid out a program to do this and began a systematic study of the finite-
time singularities. His results represented steps in the right direction, but much more
was needed. In his first preprint on the subject [11], Perelman introduced a new
ingredient, which he calls volume non-collapsing\(^6\). He showed that as singularities
develop, they are volume non-collapsed on the scale of their curvature. With this extra
condition Perelman was able to give a complete qualitative classification of models
for singularity development: every singularity at finite-time is either modeled on a
manifold of positive curvature, modeled on a manifold made up of long thin necks,
or modeled on a manifold made up of a long thin neck capped off with a 3-ball or
a punctured real projective 3-space, \(\mathbb{R}P^3\). Furthermore, Perelman established strong
analytic control on the time derivatives and the norm of the gradient of the scalar
curvature in regions of high curvature.

9. Ricci flows with surgery

Suppose that we have a 3-dimensional Ricci flow \((M, g(t)), \ a \leq t < T\), going
singular at time \(T < \infty\). In his second preprint on Ricci flow [13], Perelman extends
this flow past time \(T\) by constructing a Ricci flow with surgery. To do this he makes
use of the classification of finite-time singularities described above. At the singular
time \(T\) there is a limiting metric (possibly incomplete) defined on an open submanifold
\(\Omega \subset M\). The ends of \(\Omega\) are diffeomorphic to \(S^2 \times [0, 1]\), with metrics near any point
that look like a rescaled version of a product of a round metric on \(S^2\) with the Euclidean
metric on the interval, and with the curvature tending to infinity as one approaches the
end. Surgery consists in cutting off the ends of these tubes along one of the 2-spheres
in the product structure and sewing in a 3-ball to construct a new compact Riemannian
3-manifold \((M', g(T))\) which will be the time-slice of the Ricci flow with surgery at
time \(T\). One restarts the Ricci flow at time \(T\) using \((M', g(T))\) as the initial metric.
This flow will go singular at some time \(T' > T\). See the figure below.

\(^6\)See appendix.
It is best to view a 3-dimensional Ricci flow with surgery as a 4-dimensional space-time with a given time function $t$ to $\mathbb{R}$. There is a discrete set of singular times (in the above example there is only singular time $T$). If $T_1 < T_2$ are successive singular times, then the part of space-time, $t^{-1}([T_1, T_2])$, is simply a product of the $T$-time-slice with the interval $[T_1, T_2]$, and the Ricci flow with surgery on this part of space-time is the usual Ricci flow. As we cross a singular time both the topology and the geometry of the time-slice change, but in a controlled way as indicated above.

In [13] Perelman shows that, starting with any compact Riemannian 3-manifold, this process can be repeated forever to construct a Ricci flow with surgery defined for all positive times. Furthermore, the singular times are discrete, and the topology of the manifold before surgery is easily deduced from the topology after surgery. In particular, it is easy to see from the description of the topological change as one crosses a singular time, that if the manifold after a surgery satisfies Thurston’s geometrization conjecture, then the manifold just before that surgery also satisfies Thurston’s geometrization conjecture. Arguing by induction we see that if any time-slice of this
Ricci flow with surgery satisfies Thurston’s geometrization conjecture, then so does the initial manifold.

10. Completion of the proof of the Poincaré Conjecture

Let $M$ be a closed, simply connected 3-manifold. To prove the Poincaré Conjecture for $M$, namely to prove that $M$ is diffeomorphic to the 3-sphere, we shall show that it satisfies Thurston’s geometrization conjecture, which means that it has a Riemannian metric of constant positive curvature. From there, it is easy to see that it is diffeomorphic to the 3-sphere. Fix any Riemannian metric $g(0)$ on $M$. (Remember there is an infinite dimensional space of possibilities.) Now construct the Ricci flow with surgery defined for all time with $(M, g(0))$ as the 0 time-slice. As we noted above, if we can show that any time-slice of this Ricci flow satisfies Thurston’s geometrization conjecture, then so does $M$. So the proof of the Poincaré Conjecture is completed by showing that the time-slices of this Ricci flow with surgery at all sufficiently large times are empty, that is to say the Ricci flow with surgery becomes extinct at some finite-time, just as Hamilton showed for the Ricci flow in the case when the Ricci curvature is positive.

The special property that Perelman uses about a homotopy 3-sphere in [12] in order to prove the above finite-time extinction result is the fact that its second homotopy group vanishes and its third homotopy group is non-trivial. Associated to a non-trivial element in the third homotopy group of a Riemannian manifold is a geometric invariant. This invariant is the area of a certain disk related to the homotopy element, so that in particular the invariant is always non-negative. On the other hand, Perelman shows that as long as the Ricci flow with surgery starting with a homotopy 3-sphere does not become extinct, the derivative of this invariant is bounded above by a function that eventually becomes negative and stays bounded away from zero. This means that the invariant would have to become negative in finite-time, if the Ricci flow with surgery does not become extinct. This is impossible, showing that the Ricci flow with surgery does become extinct in finite-time.

11. Status of the Geometrization Conjecture

It seems quite likely that one can apply the existence of a Ricci flow with surgery, starting with any compact manifold, to establish the complete classification of 3-manifolds as posited by Thurston’s conjecture. For example, the arguments of the previous section apply to any prime manifold with non-trivial $\pi_2$ or $\pi_3$. The Ricci flow with surgery starting at such a manifold becomes extinct in finite time, and hence these manifolds satisfy Thurston’s conjecture. To classify all 3-manifolds, one must understand the nature of the limits as $t$ tends to infinity of the $t$ time-slices. In [13] Perelman, following similar earlier arguments of Hamilton, showed that for
all sufficiently large $t$ the time $t$ time-slice $M_t$ of a Ricci flow with surgery contains a finite collection of tori (and Klein bottles) whose fundamental groups inject into the fundamental group of $M_t$. Furthermore, the complementary components are of two general types: those of the first type admit a complete hyperbolic metric of finite volume, and those of the second type are collapsed, in the sense that they are, locally at least, metrically close to lower dimensional spaces, spaces of dimension 1 or 2. To complete the proof of Thurston’s conjecture one must show that the complementary components of the second type are unions of generalized circle bundles and 2-torus bundles where the union takes place along boundary tori whose fundamental group injects into the fundamental group of $M_t$.

Perelman has stated such a result at the end of [13], and there is a closely related result in [16], which relies on an earlier, unpublished result of Perelman. The evidence is quite strong that these arguments will withstand scrutiny, but it is still too early to say that this result has been established.

12. Future applications of Perelman’s work

Ironically enough Perelman’s proof of the Poincaré Conjecture, and even the proof of the Geometrization Conjecture, assuming that it withstands scrutiny, will have little effect of 3-dimensional topology. Those working in the subject either were already assuming these results were true, were working on hyperbolic 3-manifolds, where by definition Thurston’s conjecture holds, or were working on the properties of various algebraic, combinatorial, and geometric invariants of 3-manifolds, invariants that seem, at the present moment, to have little to do with the classification of 3-manifolds. I believe the deepest impact of Perelman’s work will lie elsewhere.

One obvious area where these ideas will have impact is in the area of 4-dimensional manifolds. We know much less about smooth 4-manifolds than we do about smooth 3-manifolds, and there are many significant questions to face before there can be full application of the Ricci flow techniques to the study of 4-manifolds. These questions include the nature of Einstein 4-manifolds (the fixed points up to scaling of the Ricci flow), as well as an extension of the curvature pinching results of Hamilton and the collapsing results of Perelman, et al., to four dimensions. Still, this is an area where there seems much possibility for future advances.

Another area, where progress is already being seen, is the area of the Kähler–Ricci flow – Ricci flow on Kähler manifolds. Here many of the analytic problems disappear, and there is much more understood about Ricci flow.

Lastly, and more speculatively, Perelman’s techniques give one strong control over singularity development in the Ricci flow for compact 3-manifolds. There are many evolution equations in both mathematics and in mathematical models of physical phenomena that are evolution equations which share many properties with Ricci flow. Understanding of singularity development in these equations could have significant influence both in mathematics and in the study of physical phenomena.
These applications, however significant, lie in the future. What we are discussing
today is a milestone – the application of Ricci flow and Perelman’s breakthroughs to
the proof of one of the most central and long-standing problems in mathematics. That
alone is enough to demonstrate the power and originality of the techniques.

Appendix. Formal definitions

The fundamental group: For any topological space $X$ with a chosen point, $x \in X$,
called the base point, the fundamental group $\pi_1(X, x)$ is defined. The elements of
this group are deformation classes of loops based at $x$. The multiplication is given by
composing loops. The identity element of the group is the class of the trivial loop at
the base point and the inverse of the class of a loop is the class of the loop with the
direction reversed.

Topological and smooth manifolds. By definition every point in a topological $n$-
manifold has a neighborhood that is homeomorphic to a neighborhood in Euclidean
$n$-space. Transferring the usual coordinates on Euclidean space by this homeomor-
phism gives us local coordinates on this neighborhood of the point. Thus, every point
in a topological manifold has a neighborhood with local coordinates like those ob-
tained by restricting the usual Euclidean coordinates to an open subset of Euclidean
$n$-space. If two coordinate systems overlap, then the coordinate functions in one
system are continuous functions of the other coordinates. A smooth manifold or
equivalently a smooth structure on a topological manifold is a choice of a subset
of coordinate systems covering the entire manifold so that when two of the chosen
coordinate systems overlap, the coordinate functions of one system are infinitely dif-
ferentiable functions of the other coordinates. Amazingly enough, Milnor’s results
show that starting in dimension 8, it is not always possible to find a smooth structure
on a topological manifold, and starting in dimension 7 it is possibly to have non-
isomorphic smooth structures on a topological manifold, in fact on the 7-sphere. The
work of Freedman [4] and Donaldson [3] show that the theory of smooth 4-manifolds
differs enormously from the theory of topological 4-manifolds.

Hyperbolic geometry. Hyperbolic space of dimension $n$ can be thought of as the
$n$-dimension sphere of radius $i$, and hence of constant curvature $-1$. The solutions to
the equation

$$-x_0^2 + x_1^2 + \cdots + x_n^2 = -1$$

in $(n + 1)$-space form a two-sheeted hyperboloid. Hyperbolic $n$-space is the upper
sheet of this hyperboloid, i.e., the intersection of this locus with $\{x_0 > 0\}$. Its group
of isometries is the subgroup of index two of the automorphism group preserving the
quadratic form $Q(x_0, \ldots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2$ preserving the two-sheets.
Hyperbolic manifolds are obtained by taking the quotient of hyperbolic space by
discrete subgroups of the isometry group that act freely on hyperbolic space.
**Homogeneous 3-dimensional manifolds.** Homogeneous 3-dimensional manifolds are by definition 3-dimensional manifolds of the form $G/H$ where $G$ is a connected Lie group and $H \subset G$ is a compact, connected subgroup. For example, $S^3$ is homogeneous because it can be written as the quotient $\text{Spin}(4)/\text{Spin}(3)$. Hyperbolic space of dimension three, is written as the quotient $\text{SL}(2, \mathbb{C})/\text{SU}(2)$. There are 9 possibilities that lead to 3-dimensional quotients. A locally homogeneous manifold is then the quotient of $G/H$ by a discrete subgroup $\Gamma \subset G$ that acts freely on $G/H$. We are only concerned with those $G/H$ that admit cofinite volume discrete subgroups $\Gamma$. This leaves only 8 three-dimensional examples. The most interesting locally homogeneous 3-manifolds are the hyperbolic ones. All others are easily classified and give fairly simple examples.

**Cutting apart 3-manifolds, Part I: The prime decomposition.** Let $X$ and $Y$ be connected, oriented 3-manifolds. The connected sum of $X$ and $Y$, denoted $X \# Y$ is formed by removing from each of $X$ and $Y$ an open 3-ball and then identifying the boundary two-spheres, to create a new 3-manifold. Notice that if $X$ is homeomorphic to the 3-sphere, then $X \# Y$ is homeomorphic to $Y$. A connected sum decomposition of a 3-manifold $M$ is an equivalence between $M$ and a connected sum $X \# Y$. By definition, it is a non-trivial connected sum decomposition if and only if neither $X$ nor $Y$ is homeomorphic to the 3-sphere. A 3-manifold is prime if it does not admit a non-trivial connected sum decomposition. It is a theorem of Kneser’s from the 1920s that every 3-manifold can be decomposed as a finite connected sum of prime manifolds, and a result due to Milnor shows that this decomposition is unique up to the order of the pieces. The first step in Thurston’s Conjecture is to decompose a general compact 3-manifold into its prime pieces. This is achieved by cutting the 3-manifold open along a finite collection of disjoint 2-spheres and capping off the resulting 2-sphere boundaries with 3-balls. Of course, in general this process will produce a disconnected manifold from a connected one.

**Cutting apart 3-manifolds, Part II: The decomposition along tori.** Thurston’s Conjecture states that every prime 3-manifold can further be decomposed along a disjoint family of tori and Klein bottles (each of which has fundamental group injecting into the fundamental group of the 3-manifold) so that each open complementary component admits a complete, locally homogeneous metric of finite volume.

**Curvature and covariant derivatives.** There is another way to view the Riemann curvature tensor. A Riemannian metric produces a way to differentiate tensors on the manifold, called covariant differentiation. If $X$ is a vector field on the manifold then covariant differentiation in the $X$ direction is denoted $\nabla_X$. Briefly, there are two types of conditions that determine the covariant differentiation associated to a metric. The first are general rules for covariant differentiation in contexts much more general than this one. They are:

1. The pairing of vector fields $(X, Y) \mapsto \nabla_X Y$ is bilinear over the scalars ($\mathbb{R}$).  

(2) The pairing in the first item is linear over the smooth functions in the first variable:
\[ \nabla_f X Y = f \nabla_X Y. \]

(3) The pairing in the first item satisfies a Leibniz rule in the second variable:
\[ \nabla_X (f Y) = f \nabla_X Y + X(f) Y, \]
where \( X(f) \) is the usual differentiation of the function \( f \) by the vector field \( X \).

The rest of the rules relate the covariant differentiation determined by a Riemannian metric to that metric and to the Lie bracket of vector fields. They are:

(4) Covariant differentiation preserves the metric in the sense that for all vector fields \( X, Y, Z \) we have
\[ X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle X, \nabla_Y Z \rangle, \]
where the brackets denote the inner product coming from the metric.

(5) The covariant derivative is symmetric in the sense that
\[ \nabla_X Y - \nabla_Y X = [X, Y], \]
where \([·, ·]\) is the usual Lie bracket of vector fields.

We then examine the extent to which the usual commutation rules fail for covariant differentiation in the coordinate directions. That is to say, suppose that we have local coordinates \((x^1, \ldots, x^n)\) on the Riemannian manifold and denote by \( \partial_i \) the (local) vector field \( \partial/\partial x^i \), in the \( i \)th-coordinate direction. Then the failure of the usual commutativity is given by
\[ R_{ij} = \nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_i}. \]
Then \( R_{ij} \) acts on vector fields so that we can form
\[ R_{ijkl} = \langle R_{ij} \partial_l, \partial_k \rangle. \]
Here, the inner product is the one determined by the metric between the pair of vector fields. Also notice that reversal of indices between the two sides of the expression. The reason for this is to make the sphere have positive rather than negative curvature. Then we have the symmetry properties of \( R_{ijkl} \): skew-symmetric in the first two variables and in the last two variables, and symmetric under interchange of variables 1 and 2 with 3 and 4.

There are many ways to view this tensor, but one of the most fruitful is to make use of all the symmetries described above and thus to consider it as a symmetric bilinear pairing on the second exterior power of the tangent bundle of the manifold. The associated quadratic form associates a real number to every two-dimensional subspace in the tangent plane to the manifold at a point. This number is the sectional curvature in the two-plane direction at the point. These sectional curvatures are the analogues of the Gauss curvature for surfaces. Of course, using the metric we...
can transform this symmetric bilinear pairing to a symmetric endomorphism of the second exterior power of the tangent bundle, and that is another useful way to view the Riemann curvature tensor. In the case of a surface, the Riemann curvature tensor is equivalent to the Gauss curvature.

**Volume non-collapsing.** Suppose that we have a Ricci flow \((M, g(t))\) on an \(n\)-dimensional manifold, and a point \(p \in M\). The curvature scale at \((p, t)\) is the largest \(r > 0\) such that the norm of the Riemann curvature tensor is bounded by \(r^{-2}\) on the ball of radius \(r\) in \((M, g(t))\) centered at \(p\), denoted \(B(p, t, r)\), for all the metrics \(g(t')\) for \(t' \in [t - r^2, t]\). Fix a positive constant \(\kappa\). We say that \((M, g(t))\) is \(\kappa\)-non-collapsed on the scale of its curvature at \((p, t)\) if for \(r\) equal to the curvature scale at \((p, t)\) we have \(\text{Vol} B(p, t, r) \geq \kappa r^n\).

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